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PROJECTIONS OF THE TWISTED CUBIC

Joerg Meyer

Abstract. One gets every cubic curve with rational parametrization by projecting the curve with general point $(t \mid t^2 \mid t^3)$ in different ways onto planes. This result shows the power of elementary methods.

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0. Introduction

The space curve having the general point $K(t) = (t \mid t^2 \mid t^3)$ with $t \in \mathbf{R}$ is called *twisted cubic*. Fig. 1 shows three different parallel orthographic projections ("orthographic" means that the direction on the projections is perpendicular to the viewplane).

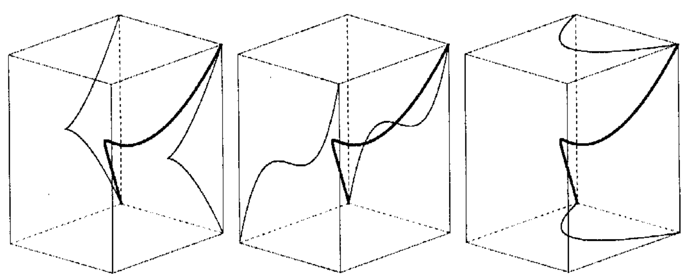


Fig. 1

part of Fig. 1	equation of viewplane	projected point	equation of curve	name of curve
left	$x = \pm 1$	$(\pm 1 \mid t^2 \mid t^3)$	$y^3 = z^2$	Neil's parabola
center	$y = \pm 1$	$(t \mid \pm 1 \mid t^3)$	$z = x^3$	cubic parabola
right	$z = \pm 1$	$(t \mid t^2 \mid \pm 1)$	$y = x^2$	quadratic parabola

The projections in Figure 1 may induce the following question:

- Is it possible to get any cubic curve (with rational parametrization) by an appropriate projection (parallel or central) of the twisted cubic?

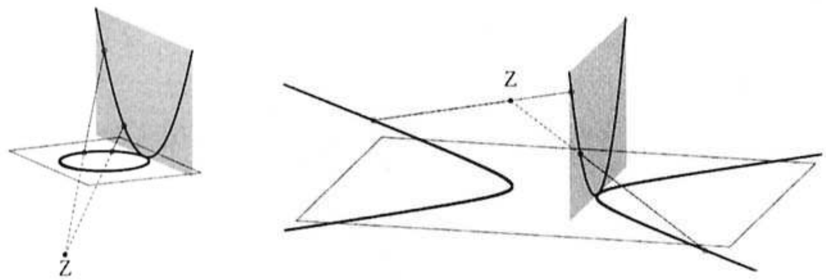


Fig. 2

We will answer that question in the affirmative. This generalizes the following well-known fact about conics: *Any quadratic curve can be obtained by a central projection of the quadratic parabola* (Fig. 2).

If the parabola is given by the general point $P(t) = (1 \mid t \mid t^2)$, the central projection from $Z = (0 \mid 0 \mid \varepsilon)$ onto the plane with $z = 0$ maps $P(t)$ to $\dfrac{\varepsilon}{\varepsilon - t^2} \cdot (1 \mid t \mid 0)$. This is an ellipse for $\varepsilon = -1$ and a hyperbola for $\varepsilon = 1$.

On the other hand, there is no hope of getting a *general* plane cubic curve by projecting the twisted cubic because those projections result in plane curves with the general point $\begin{pmatrix} f(t)/h(t) \\ g(t)/h(t) \end{pmatrix}$ where f, g and h are polynomials. This shows that only cubic curves with rational parametrization may be obtained in this way.

An example of a cubic curve with no rational parametrization is given by the Fermat curve with $x^3 + y^3 = 1$ (that curve has only two points with rational coordinates).

This paper assumes some familiarity with affine mappings and projective concepts (e.g., see [1]). Basic projective notions will be collected in section 2.

1. Other parallel orthographic projections

Let us first look at orthographic projections parallel to the vector $N = (1 \mid 0 \mid c)$. The curve point $K = (t \mid t^2 \mid t^3)$ is mapped to

$$Q = K - \frac{K \cdot N}{N^2} \cdot N = t^2 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \frac{t^3 - c \cdot t}{1 + c^2} \cdot \begin{pmatrix} -c \\ 0 \\ 1 \end{pmatrix}.$$

Since $(0 \mid 1 \mid 0)$ and $(-c \mid 0 \mid 1)$ make up a local orthogonal system in the viewplane, Q can be written as $Q = \begin{pmatrix} t^2 \\ t^3 - c \cdot t \end{pmatrix}$, neglecting scalings. Figure 3 shows the curves for $c = 1$ and $c = -1$.

Isaac Newton called the curves with general point $\begin{pmatrix} t^3 - c \cdot t \\ t^2 \end{pmatrix}$ *divergent parabolas* ([3]; all the names of the following cubic curves are due to him). In Newton's

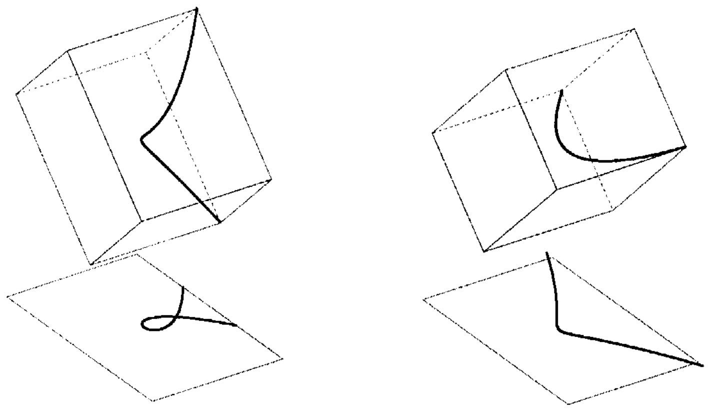


Fig. 3

classification, there are two more types of divergent parabolas, but those do not have a rational parametrization.

Divergent parabolas with $\begin{pmatrix} t^3 - c \cdot t \\ t^2 \end{pmatrix}$ with $c \neq 0$ can also be obtained by a central projection from $Z = (0 \mid -1/c \mid 0)$ onto the plane with $z = 1$. The result of the mapping is

$$\begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix} + \frac{1-t^3}{t^3} \cdot \begin{pmatrix} t \\ t^2 - 1/c \\ t^3 \end{pmatrix} = \begin{pmatrix} 1/t^2 \\ 1/c + (t^2 - 1/c)/t^3 \\ 1 \end{pmatrix}.$$

If one neglects translation and scalings and changes the parameter from t to $1/s$ the result is $(s^2 \mid s^3 - c \cdot s \mid 1)$.

2. Newton's idea of how to classify the cubic curves

How can all the cubic curves (cubic parabolas, divergent parabolas and maybe a lot more) be classified? It is a good idea to mimic the classification of conics.

An affine treatment regards two curves as equivalent if there is an affine mapping between them, so it disregards scalings and shears. Then there are three (nondegenerate) types: ellipse, parabola and hyperbola. That shears are disregarded means that the curves with $y = x^3$ and $y = x^3 - x$ are considered to be equivalent, even though their numbers of different zeros differ.

Ellipse, parabola and hyperbola differ in the number of different intersections with the line at infinity. In the sequel, this important line will be abbreviated as u.

The line u can be visualized by a well-known procedure: Consider the (affine) point $(x \mid y) \in \mathbf{R}^2$ as a point $P := (x \mid y \mid 1) \in \mathbf{R}^3$ lying in the plane with $z = 1$.

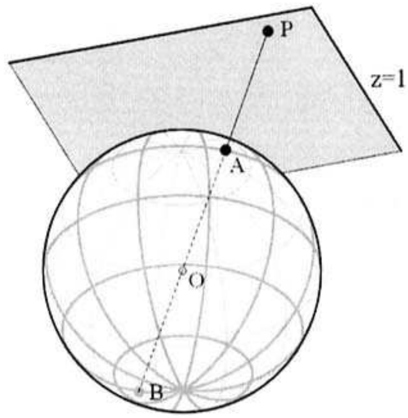


Fig. 4

The associated projective point is the line through

$$O := (0 \mid 0 \mid 0)$$

and P; it has homogeneous coordinates $(x : y : 1)$. The line through O and P intersects the sphere with $x^2 + y^2 + z^2 = 1$ in the antipodal points A and B (Fig. 4). The "points at infinity" have coordinates $(x : y : 0)$ and are mapped onto the equator line of the sphere. Two special points on u are $U := (0 : 1 : 0)$ and $V := (1 : 0 : 0)$; they will occur many times in the sequel.

Figure 5 shows the projections of the parabola and the hyperbola. Only one projection is shown. The parabola (considered to live in projective space) intersects u with multiplicity 2, the hyperbola intersects u in two distinct points, the ellipse does not intersect u at all.

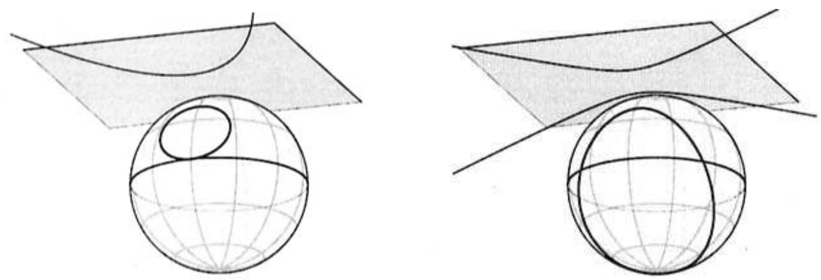


Fig. 5

These intersection properties can also be computed:

For the parabola we have

$$\begin{pmatrix} t \\ t^2 \end{pmatrix} \cong (t : t^2 : 1) \overset{t=1/s}{=} (s : 1 : s^2),$$

and for $s = 0$ (with multiplicity 2) we get U as point at infinity.

For the hyperbola, the analogous computation is

$$\left(t : \frac{1}{t} : 1\right) = (t^2 : 1 : t) \overset{t=1/s}{=} (1 : s^2 : s).$$

We get two distinct points U and V at infinity for $t = 0$ and for $s = 0$.

How does the cubic parabola behave at infinity? Because of

$$\begin{pmatrix} t \\ t^3 \end{pmatrix} \cong (t \ : \ t^3 \ : \ 1) \overset{t=1/s}{=} (s^2 \ : \ 1 \ : \ s^3),$$

we get for $s = 0$ an intersection with u at U with multiplicity 3 (Fig. 6).

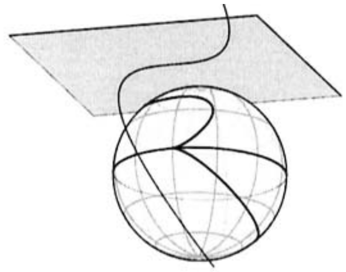


Fig. 6

The divergent parabolas behave at infinity in a uniform way which is different from that of the cubic parabola. Figure 7 shows

$$\begin{pmatrix} t^2 \\ t^3 - c \cdot t \end{pmatrix} \cong (t^2 \ : \ t^3 - c \cdot t \ : \ 1) \overset{t=1/s}{=} (s \ : \ 1 - c \cdot s^2 \ : \ s^3)$$

for $c = 1$, $c = 0$, and $c = -1$.

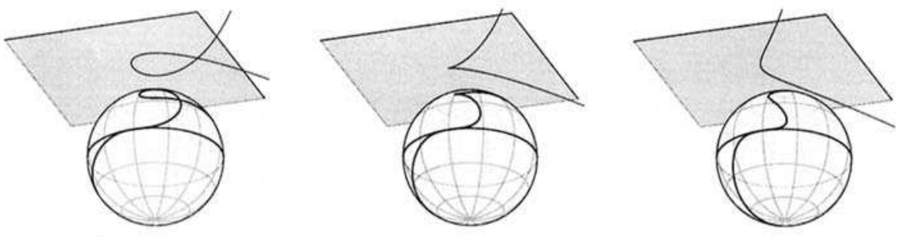


Fig. 7

The question arises: How to distinguish between Figure 7 and Figure 6?

3. How to distinguish between different types of tangents

The cubic parabola has the general point $(s^2 \ : \ 1 \ : \ s^3)$; for $s = 0$ we get $U = (0 \ : \ 1 \ : \ 0)$. Now, every line in the projective plane has an equation of the type $a \cdot x + b \cdot y + c \cdot z = 0$. So any line through U has the form $a \cdot x + c \cdot z = 0$. If we are going to intersect this general line with the curve we get the equation of intersection

$$a \cdot s^2 + c \cdot s^3 = s^2 \cdot (a + c \cdot s) = 0.$$

That equation shows: *Any* line through U intersects the curve with multiplicity ≥ 2. (Those points are called *non-regular.*) For the line with $a = 0$ (i.e., with the equation $z = 0$) we have multiplicity > 2. So u is tangent to the curve.

The situation is different for divergent parabolas: Setting $s = 0$ in the general point $(s \ : \ 1 - c \cdot s^2 \ : \ s^3)$, we get U again. The equation of intersection between an arbitrary line through U (with equation $a \cdot x + c \cdot z = 0$) and the curve now is

$$a \cdot s + c \cdot s^3 = s \cdot (a + c \cdot s^2) = 0.$$

Any line through U intersects the curve with multiplicity ≥ 1. (So U is *regular.*) For the line with $a = 0$ we have multiplicity > 1. So u is tangent to the curve.

4. How many intersections does a cubic curve have with u?

It may be a good idea to carry over the classification scheme from conics to cubics: Two cubics are considered to be equivalent if they intersect with u in the same way. So we have to ask: How does the curve behave at u? How many intersections are there, and what about the multiplicities and non-regular points?

Any cubic curve has a general point $(f(t) : g(t) : h(t)) \overset{t=1/s}{=} (\bar{f}(s) : \bar{g}(s) : \bar{h}(s))$ in homogeneous coordinates with polynomials f, g and h of degree $\leqslant 3$. Intersecting the curve with u one gets the equation

$$h(t) = 0 \quad \text{or} \quad \bar{h}(s) = 0.$$

These equations always have three zeros which do not have to be distinct ones. This is obvious if the degree of h or the degree of $\bar{h}$ equals 3. But it is also true if h and $\bar{h}$ are both quadratic. E.g., for $h(t) = 1 + t^2$ we have $\bar{h}(s) = s^2 + s$, and the intersection with u are produced by $t = 0$, $t = -1 = s$, and $s = 0$.

So we have the result: *Any cubic curve intersects u in at least one point and in at most three points.* The intersecting point which is there in any case, shall be U. The multiplicity of u and the curve at U is denoted by μ. The following figures do not contain the plane with $z = 1$ anymore.

5. Cubic curves having $\mu = 3$

There are three possibilities:

5.1. U is regular and u is the only tangent. This is the case with *divergent parabolas* (Fig. 7).

5.2. U is non-regular with u as only tangent. This is the case with *cubic parabolas* (Fig. 6).

5.3. The line u is not the only tangent in U. This is only possible if U is a double point. That case is new and deserves special attention.

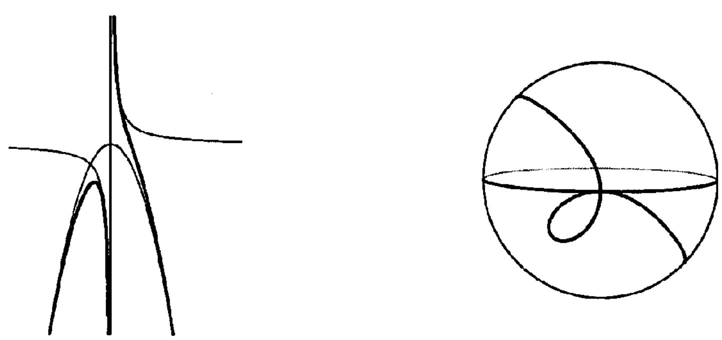


Fig. 8

The general point could be

$$(t \;:\; t^3 - 1 \;:\; t^2) \overset{t=1/s}{=} (s^2 \;:\; 1 - s^3 \;:\; s).$$

For $t = 0$ we get U. An arbitrary line through U has the form $a \cdot x + c \cdot z = 0$. The equation of intersection is $t \cdot (a + c \cdot t) = 0$. For $a = 0$ we have multiplicity $= 2$, so u is a tangent.

For $s = 0$ we get U again. In this case the equation of intersection is

$$s \cdot (a \cdot s + c) = 0.$$

For $c = 0$ we have multiplicity $= 2$, so the line with $x = 0$ is a tangent at U as well.

The curve with this behavior is called *trident*. Figure 8 shows the curve in relation to u.

The left part of Fig. 8 shows the affine part of the trident (without the coordinate axes). For $t = 0$ we get the osculating conic with general point $(t \;:\; -1 \;:\; t^2) = \left(\frac{1}{t} \;:\; -\left(\frac{1}{t}\right)^2 \;:\; 0\right)$, a parabola. For $s = 0$ we get the osculating conic with general point $(s^2 \;:\; 1 \;:\; s) = (s \;:\; \frac{1}{s} \;:\; 1)$, a hyperbola. The osculating line for $s = 0$ has the general point $(0 \;:\; 1 \;:\; s)$ and the equation $x = 0$.

Now we have to show that the above trident can be obtained by appropriate central projections of the twisted cubic. Consider the projection from $Z = (0 \mid 0 \mid 1)$ onto the plane with $y = 1$: it maps $(t \mid t^2 \mid t^3)$ to $(1/t \mid 1 \mid 1 + (t^3 - 1)/t^2)$; in that plane, neglecting translations we have

$$\begin{pmatrix} 1/t \\ (t^3-1)/t^2 \end{pmatrix} \cong \left(\frac{1}{t} \;:\; \frac{t^3-1}{t^2} \;:\; 1\right) = (t \;:\; t^3 - 1 \;:\; t^2).$$

6. Cubic curves having $\boldsymbol{\mu} = 2$

In this case there has to be another intersection between the curve and u. We choose the other intersection to be $V = (1 \;:\; 0 \;:\; 0)$. Figure 9 shows what can happen.

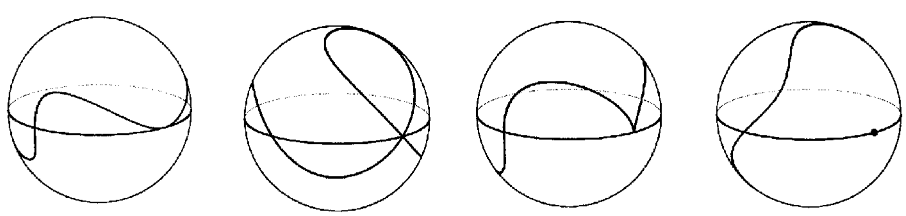


Fig. 9

Here are examples for the four possibilities.

6.1. The line u is tangent at U. A possible general point is

$$(s^2+1 \ : \ s^3 \ : \ s) = (t+t^3 \ : \ 1 \ : \ t^2) \cong \begin{pmatrix} t+1/t \\ 1/t^2 \end{pmatrix}.$$

For $s=0$ we get V and for $t=0$ we get U. A general line through U has the form $a \cdot x + c \cdot z = 0$ and leads to the equation of intersection

$$t \cdot (a \cdot (1+t^2) + c \cdot t) = 0.$$

We see that the tangent is determined by $a=0$, so u is the tangent at U.

A curve of this type is called *parabolic hyperbola*. Parabolic hyperbolas can have quite different affine shapes. This is true for each of the following types as well and won't be mentioned anymore. One possible affine shape is shown in Fig. 10, other ones will be referred to in the last chapter.

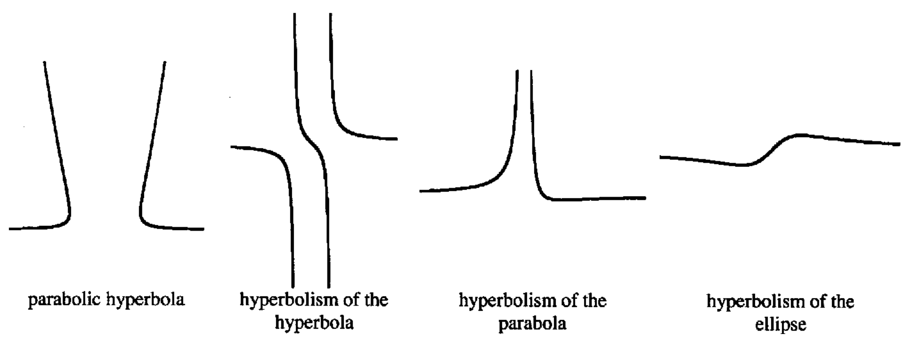


Fig. 10

6.2. U is double point and both tangents are real and distinct. An example is given by $(t^3-t \ : \ t \ : \ t^2-1) = (1-s^2 \ : \ s^2 \ : \ s-s^3)$. For $t=\pm 1$ we get U and for $s=0$ we get V.

A general line through U with $a \cdot x + c \cdot z = 0$ induces the equation of intersection

$$(t-1)(t+1) \cdot (a \cdot t + c) = 0.$$

For $t = \pm 1$, the tangent is given by $a = \mp c$. Curves with a double point on u and having real distinct tangents are called *hyperbolisms of the hyperbola*.

6.3. U is a cusp and u is not a tangent. One example is given by the general point $(t^3 \ : \ 1-t \ : \ t^2) = (1 \ : \ s^3-s^2 \ : \ s)$. A general line through U leads to the equation of intersection

$$a \cdot t^3 + c \cdot t^2 = t^2 \cdot (a \cdot t + c) = 0.$$

Any line through U cuts u with $\mu = 2$. For $c=0$ we get $\mu = 3$, so the line with $x=0$ is the tangent to U.

Curves with a cusp on u without being the tangent there are called *hyperbolisms of the parabola*.

6.4. U is an isolated point. A general point is $(t+t^3 \;:\; t \;:\; t^2+1) = (s^2+1 \;:\; s^2 \;:\; s+s^3)$. For the non-real values $t = \pm\sqrt{-1}$ we get the real point U and for $t = 0$ we get V.

Curves which have an isolated point on u are called *hyperbolisms of the ellipse*.

All these curves (or their shears) can be obtained by an appropriate central projection.

number	name of the curve	viewplane	Z	mapped point in the viewplane, neglecting translations and scalings
6.1	parabolic hyperbola	$x=1$	$\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} (t^2+1)/t \\ t^3/t \end{pmatrix} \cong (t^2+1 \;:\; t^3 \;:\; t)$
6.2	hyperbolism of the hyperbola	$y=-1$	$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} t/(t^2-1) \\ t^3/(t^2-1) \end{pmatrix} \cong (t \;:\; t^3 \;:\; t^2-1)$
6.3	hyperbolism of the parabola	$y=1$	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} (t-1)/t^2 \\ t \end{pmatrix} \cong (t-1 \;:\; t^3 \;:\; t^2)$
6.4	hyperbolism of the ellipse	$y=1$	$\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} t/(t^2+1) \\ t^3/(t^2+1) \end{pmatrix} \cong (t \;:\; t^3 \;:\; t^2+1)$

7. Cubic curves having $\mu = 1$

Here two cases are possible (Fig. 11):

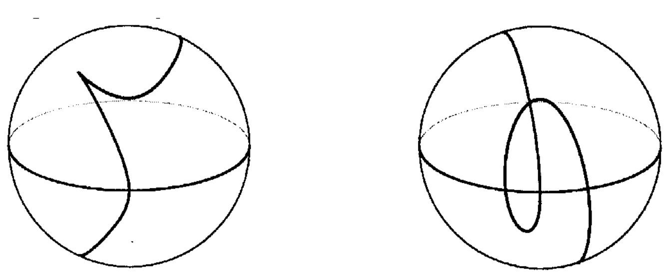


Fig. 11

7.1. U is the only intersection (with $\mu = 1$) between the curve and u. A possible general point is $(t^3-t^2 \;:\; 1 \;:\; t^3+t) = (1-s \;:\; s^3 \;:\; 1+s^2)$. For $t=0$ we get U. But for $s = \pm\sqrt{-1}$ we get the non-real point $(\pm 1 \;:\; \sqrt{-1} \;:\; 0)$. A curve with this behavior at u is called a *defective hyperbola*. The affine picture is shown in Fig. 12.

7.2. The curve intersects u in three different point, one of those is U. A possible general point is $(t^2 : 1-t : t^3-t) = (s : s^3-s^2 : s^2-1)$. Such a curve is called a *redudant hyperbola*.

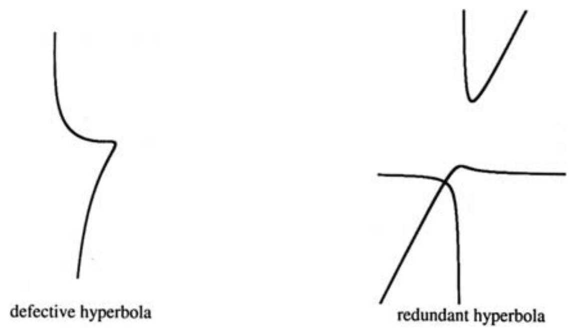


Fig. 12

These curves arise as central projections of the twisted cubic, too:

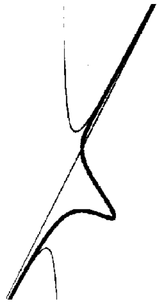
number	name of the curve	viewplane	Z	mapped point in the viewplane, neglecting translations and scalings
7.1	defective hyperbola	$y=1$	$\begin{pmatrix}1\\-1\\0\end{pmatrix}$	$\begin{pmatrix}(t-1)/(t^2+1)\\t^3/(t^2+1)\end{pmatrix} \cong (t-1 : t^3 : t^2+1)$
7.2	redudant hyperbola	$y=-1$	$\begin{pmatrix}0\\1\\1\end{pmatrix}$	$\begin{pmatrix}t/(t^2-1)\\(t^3-t^2)/(t^2-1)\end{pmatrix} \cong (t : t^3-t^2 : t^2-1)$

8. Conclusion

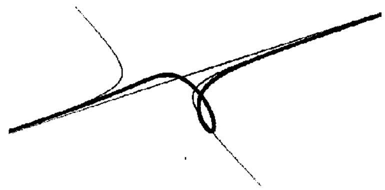
We have seen that one obtain a cubic curve (with rational parametrization) of any type of behavior at u as an appropriate projection of the twisted cubic.

9. Different affine shapes

It is a nice exercise to generate affine shapes of the curves mentioned above. To stimulate the imagination, the following figures show some examples together with their linear and conical asymptotes.

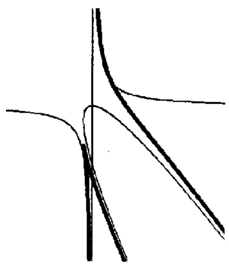


hyperbolism of the ellipse
viewplane: $z = 0$, $Z = (0 \mid -1 \mid 1)$

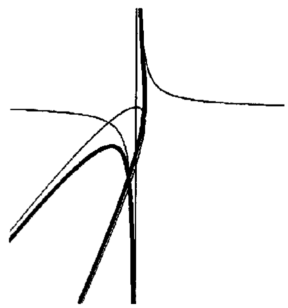


defective hyperbola
viewplane: $z = 0$, $Z = (-0.5 \mid 0.5 \mid 1)$

Fig. 13

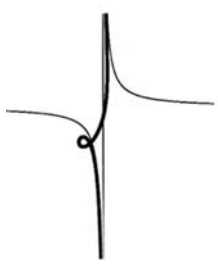


parabolic hyperbola
viewplane: $y = 0$, $Z = (-0.6 \mid 0 \mid 1)$

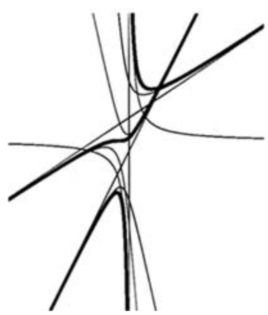


parabolic hyperbola
viewplane: $y = 0$, $Z = (0.6 \mid 0 \mid 1)$

Fig. 14

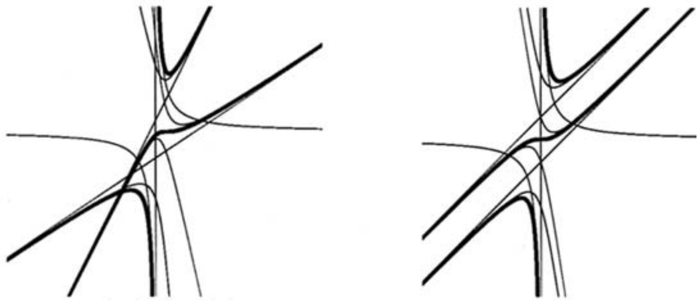


defective hyperbola
viewplane: $y = 0$, $Z = (1 \mid -1 \mid 2)$



redundant hyperbola
viewplane: $y = 0$, $Z = (-0.5 \mid 1 \mid 0)$

Fig. 15



redundant hyperbola
viewplane: $y = 0$, $Z = (0.5 \mid 1 \mid 0)$

hyperbolism of the hyperbola
viewplane: $y = 0$, $Z = (0 \mid 1 \mid 0)$

Fig. 16

trident
viewplane: $x = 0$, $Z = (1 \mid 1 \mid 2)$

parabolic hyperbola
viewplane: $x = 0$, $Z = (1 \mid 2 \mid 0)$

Fig. 17

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Dr Joerg Meyer, Schaefertrift 16, D-31789 Hameln, Germany
E-mail: `J.M.Meyer@t-online.de`