THE TEACHING OF MATHEMATICS

2011, Vol. XIII, 1, pp. 45–55

# TEACHING ENVELOPES IN SECONDARY SCHOOL

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**Abstract.** The paper presents an educational module, in the spirit of inquiry based method, which is concerned with the case of some simple envelopes. The inclusion of dynamic geometry system applets in presenting the topic allows the shape of the envelope of one-parameter family of curves to be established by examining dynamic constructions. Starting with an example for the Euler formula, further motivation of introduction of envelopes belongs to the field of kinematics. Two main problems are under consideration: the misty sprinkler and the wet wheel. The introductory part of the kinematics problems does not go out of the common school practice.

*ZDM Subject Classification*: U54; *AMS Subject Classification*: 97U50.

*Key words and phrases*: Dynamic geometry system; parabola; envelopes.

## 1. Introduction

The modern computer technologies allow to present some mathematical ideas that require higher mathematics in a way suitable for secondary school students. One topic of this type is the envelope of a family of curves. Technically this was not easy to be done in the pre-computer era: it was a serious didactical challenge to illustrate in real time what one-parameter family of curves looks like, e.g. to draw a picture (like that on Fig. 1, [1]), while a lesson runs. Nowadays the illustrative part of the matter is 'a peace of cake' but another didactical challenge still stands on agenda: to motivate introduction of the concept "envelope". Below we are going to overcome the last challenge laying a bridge to kinematics.

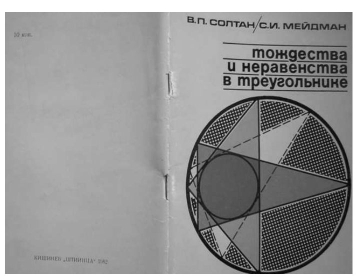


Fig. 1

The target group we have in mind is 10th–12th grade students particularly advanced in mathematics and science. A priori time consumption is about 90 minutes, which corresponds to the single extracurricular lesson but it strongly depends on the classroom context and could be considerably extended. Students are expected to be acquainted with the idea of loci, Pythagorean Theorem, 2D coordinates, quadratic systems, algebraic inequalities, parametric presentation as well as to have basic knowledge in kinematics and initial skills in dynamic geometry systems (e.g. GEONExT or GeoGebra).

## Starting point

It is always easier to start a new topic with something that students are already familiar with, so we are going to give an idea how envelopes appear, considering a family of line segments whose envelope is a circle. The picture on Fig. 1 could be obtained performing by dynamic geometry system (DGS) the next construction:

- draw two circles $K$ and $k$ with radii 9 and 4 and centered in points at distance 3,
- take an arbitrary point $A$ on $K$ and lay the tangents from $A$ to $k$,
- denote by $B$ and $C$ the intersection points of these tangents and $K$,
- draw the segment $BC$ and turn on the trace function.

Now moving $A$ along $K$ students can explore the shape of the set of points inside $K$ none of which lay on any segment $BC$. More general is the following

Problem 1. (Dynamized *Poncelet Theorem*) Given two segments $R$ and $r$, such that $R > 2r$, construct two circles $K$ and $k$ with radii $R$ and $r$ respectively and centered at points $\sqrt{R(R-2r)}$ apart each other. Take an arbitrary point $A$ on $K$ and lay the tangents from $A$ to $k$. Let $B$ and $C$ be the intersection points of tangents and $K$.

a) Moving $A$ along $K$ explore the shape of the set of points inside any triangle $ABC$ which do not lie on its sides.

b) Justify the observation done in a).

*Comment.* The justification of b) is direct implementation of the Euler Formula $\delta^2 = R^2 - 2Rr$ connecting the radii of circumcircle and incircle with the distance $\delta$ between centers. This formula is included in the 10th grade math curriculum in Bulgaria. Some exploration work in Problem 1 could be organized examining DGS parametric constructions. For instance the next GeoGebra-construction-protocol is for the construction on Fig. 2:

- construct two circles $K$ and $k$ with radii $R$ and $r$ respectively and with the distance $d$ between their centers,
- take an arbitrary point $A$ from $K$ and lay the tangents to $k$ through $A$,
- take the intersection points $B$ and $C$ of the tangents and $K$,

| No. | Name | Definition | Value |
|---|---|---|---|
| 1 | Point O | | O=(-1.3,1.88) |
| 2 | Number d | | d=1 |
| 3 | Point P | Point on Circle[O,d] | P=(-0.3,1.88) |
| 4 | Segment $d_1$ | Segment[O,P] | $d_1$=1 |
| 5 | Number R | | R=5 |
| 6 | Number r | | r=2 |
| 7 | Circle K | Circle with center O and radius R | K: $(x+1.3)^2+(y-1.88)^2=25$ |
| 8 | Circle k | Circle with center P and radius r | k: $(x+0.3)^2+(y-1.88)^2=4$ |

| | | | |
|---|---|---|---|
| 9 | Point A | Point on K | A=(2.97,-0.71) |
| 10 | Line c | Tangent to k through A | c: -3.38x-1.43y=-9.02 |
| 10 | Line b | Tangent to k through A | b: -0.62z-3.61y=0.73 |
| 11 | Point $B_1$ | Intersection point of K, c | $B_1$=(2.97,-0.71) |
| 11 | Point B | Intersection point of K, c | B=(-0.2,6.76) |
| 12 | Point $C_1$ | Intersection point of K, b | $C_1$=(2.97,-0.71) |
| 12 | Point C | Intersection point of K, b | C=(-6.2,0.87) |
| 13 | Segment a | Segment[B,C] | a=8.41 |

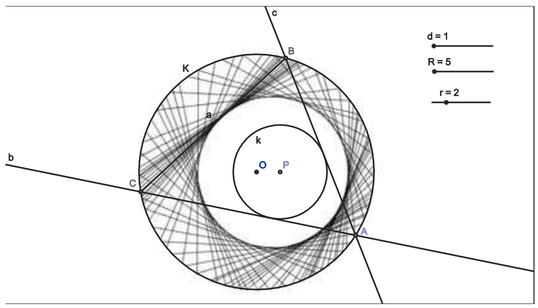


Fig. 2

Moving $A$ along $K$ on the screen appears the envelope of the family of segments $BC$ that looks like a circle (call it $\kappa$). There are two other envelopes: of family lines $AB$ and $AC$ which by definition coincides with $k$ and could be observed coloring $AB$ and $AC$ in different colors. Changing parameters $r$ and $d$ students can observe that the envelope $\kappa$ coincides with $k$ if the parameters satisfy the Euler formula. They can argue that if is in fact iff.

### Tangency definition troubles

The general definition of the envelope of a family of curves says that it is *a curve that touches each curve of the family at some point*. We do not know any other suitable way to define directly touching curves in the frame of the secondary school curriculum. So we are going to define touching curves in two steps:

1) first, we will define for any curve what is a line to be tangent to the curve at one of its points;
2) we say that *two curves touch each other* in one of their common points if they have common tangent at that point.

The above two-steps procedure is embarrassing because passing through a touching point, the order of touching of curves is lost. However, for our goals this defect does not seem to be much relevant.

Definition. We say that *a line touches parabola* if the line and the parabola have just one common point. The line is called tangent to this parabola.

*Comment*. Any parabola is the envelope of the perpendicular bisectors of the segments having one end in the focus of the parabola and the other end on its directrix. In fact, the most important property of the parabola—the reflective property—is a corollary of the presentation of the parabola as the just described envelope: *the parabola reflects the incident rays in the way the envelope of its tangents does* [2]. The above definition of tangent to parabola corresponds to the definition of tangent to circle. Further extension of defining tangent in the same way should be done carefully because of the local matter of touching, e.g. the abscissa touches the cubic parabola $y = x^3 - x^2$ in the origin but has another common point $(1, 0)$ at which the abscissa crosses this cubic parabola.

### Envelope of a family of parabolas

PROBLEM 2. Determine the set of points in the plane that are not lying on any of the parabolas

$$y = x^2 - 2px + 2p^2 - 3, \quad p \in (-\infty, +\infty).$$

*Solution* [3]. Let $X(x, y)$ be a point that does not lie on any of the given parabolas. Then

$$y \neq x^2 - 2px + 2p^2 - 3, \quad \forall x \in (-\infty, +\infty),$$

i.e. the following quadratic equation with respect to $p$

$$2p^2 - 2xp + x^2 - y - 3 = 0$$

has no real roots. Thus, in order to determine the desired set of points, we have to inquire the discriminant of the above quadratic equation in terms of $(x, y)$ considered as parameters:

$$\frac{1}{4}D = x^2 - 2(x^2 - y - 3).$$

The inequality $D < 0$ is equivalent to

$$y < \frac{1}{2}x^2 - 3,$$

which is the condition for $X(x, y)$ to be a point not lying on any of the parabolas: the desired set consists of the points below (i.e. outside) the parabola $y = \dfrac{1}{2}x^2 - 3$.

According to the common definition of touching curves we say that *two parabolas touch each other* in a point if they have common tangent at this point.

*Challenge*. Justify that the parabola $y = \dfrac{1}{2}x^2 - 3$ touches any of the parabolas $y = x^2 - 2px + 2p^2 - 3$, $p \in (-\infty, +\infty)$, i.e. that the simultaneous equations

$$\begin{aligned} y &= \frac{1}{2}x^2 - 3, \\ y &= x^2 - 2px + 2p^2 - 3 \end{aligned}$$

have exactly one solution $(x_*, y_*)$ for any $p \in (-\infty, +\infty)$ and both parabolas have common tangent at $(x_*, y_*)$.

Corollary. The parabola $y = \frac{1}{2}x^2 - 3$ touches any of the given parabolas, i.e. it is the envelope of the family of the given parabolas (Fig. 3.).

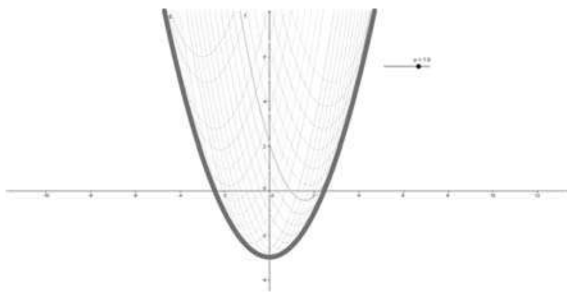

Fig. 3

*Dynamic drill.* Visualize the envelope by DGS.

The next problem describes how a facility works. It could be seen in parks, atriums etc. Pavel Boychev created a short movie (following the instructions of the author) to illustrate the sprinkler and Fig. 4 is a screenshot from it [4].

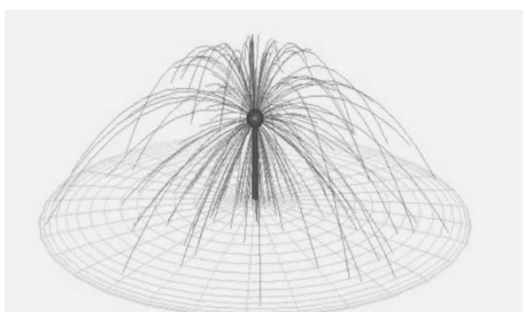

Fig. 4

Problem 3. (*The misty sprinkler.*) Water dust is produced by a device with point-tiny head that sprinkles water particles uniformly in any direction producing any particle initial speed with constant magnitude. Determine the shape of the cloud neglecting the air resistance.

*Solution.* Let us take the origin of the (Cartesian) coordinate system at the source of the dust. Consider a slice of the cloud by the vertical plane through the origin and which contains the abscissa. Let a water particle $W$ leaves the source in the moment having in this moment $t = 0$ the vector of velocity

$$\vec{v} = (q\cos\varphi, q\sin\varphi), \qquad \varphi \in [0, 2\pi).$$

Here $q$ is the magnitude of the initial speed of the particle and depends on the technical characteristics of the device which do not change during the period of our observation; $\varphi$ is the angle between the abscissa and the direction in which the device shoots the particle $W$. The coordinates of $W$ in the moment $t > 0$ are

$$x(t) = (q\cos\varphi)t$$
$$y(t) = (q\sin\varphi)t - \frac{1}{2}gt^2$$

($g$ stands for the acceleration of gravity).

From the system we find

$$y = -\frac{g}{2q^2\cos^2\varphi}x^2 + \tan\varphi x.$$

Let $p = \tan\varphi$. Then

$$y = -\frac{g}{2q^2}(p^2+1)x^2 + px, \quad p \in (-\infty, +\infty).$$

As in the preceding problem we consider the above equation to be a quadratic with respect to $p$:

$$(gx^2)p^2 + 2q^2x \cdot p + (gx^2 + 2q^2y) = 0.$$

A point $Q(x, y)$ avoids any wet trajectory iff its coordinates do not satisfy the above equation for any $p$. This happens when the discriminant

$$D = 4q^4x^2 - 4qx^2(gx^2 + 2q^2y)$$

is negative. The inequality $D < 0$ is equivalent to

$$y > \frac{q^2}{2g} - \frac{g}{2q^2}x^2,$$

which is the condition for $Q(x, y)$ to be a dry point: the shape of the cloud is formed by the parabolas

$$y = \frac{q^2}{2g} - \frac{g}{2q^2}x^2$$

under revolution about the vertical axis.

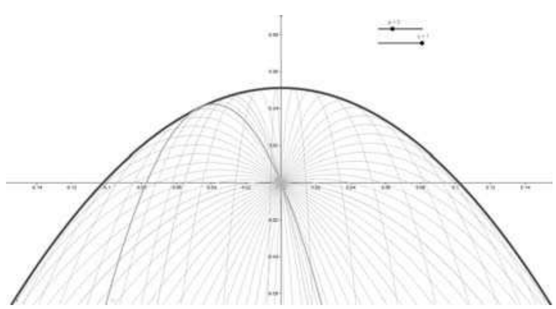

Fig. 5

*Dynamic drill.* Visualize the result from Problem 3 by DGS.

Fig. 5 shows a plane slice of the wet area produced by the misty sprinkler—obviously this area is the envelope of the family of water particle trajectories parameterized with $\varphi$. Here we assume $q$ is fixed for all trajectories, so the envelope is parameterized with $q$.

*Challenge.* Explore the one-parameter parabola family of the envelopes of the wet area of the misty sprinkler.

**Envelope of a family of circles**

*Dynamic drill.* Draw a parabola $\Pi$ with axis $a$ and take an arbitrary point $P$ on it. Lay the tangent $b$ through $P$ to $\Pi$ and erect the perpendicular $n$ from $P$ to $b$. Let $C$ be the intersection point of $a$ and $n$. Draw the circle $c$ centered at $C$ and with radius $r = CP$. Explore the construction by moving $P$ along $\Pi$.

According to the common definition of touching curves we say that *circle touches parabola* in a point if the circle and the parabola have common tangent at this point.

*Comment.* It is obvious that $b$ is tangent to $c$ in the above construction, i.e. $c$ touches $\Pi$. Now note that any circle $c$ (except one) has two common points with $\Pi$. Hence the "definition" *a circle touches a parabola if the both curves have exactly one common point* does not work. It should be modified in a way that takes into account the configuration of the curves in a *neighborhood* of their common point.

*Challenge.* Draw a parabola and a circle which is touching but also crossing this parabola.

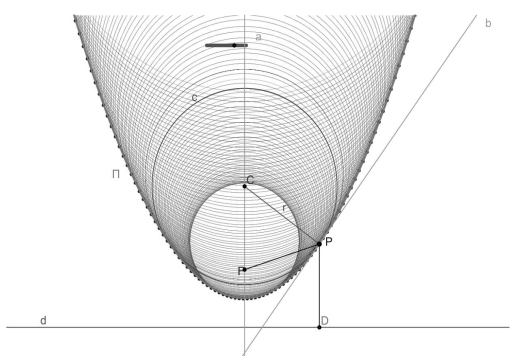


Fig. 6

Fig. 6 presents $\Pi$ as an envelope of one-parameter family of circles. Let us note that we are not in a vicious circle: we have constructed any $c$ as touching $b$, but the line $b$ could be defined independently of $\Pi$ [2]. This envelope could be obtained in our dynamic construction after turning on the trace option for $c$.

*Comment.* Any of the circles $c$ approximates the parabola $\Pi$ near the point of tangency but it is not the best approximation of $\Pi$. The best approximation of

$\Pi$ in any particular point is given by another circle (which defines the curvature of the parabola) but $\Pi$ is not an envelope of the family of these best circles.

The next problem describes how a wet wheel sprinkles surrounding area. Fig. 7 is a screenshot from another short movie done for us by Pavel Boychev [5].

PROBLEM 4. (*The wet wheel* [6]). A wet wheel rotates in a vertical plane sprinkling the neighboring area. Determine the dry area in the plane.

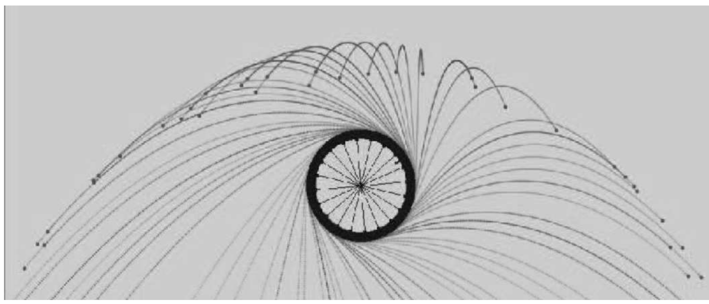

Fig. 7

*Solution.* Let us take the origin of the (Cartesian) coordinate system at the center of the wheel and let the abscissa be horizontal.

Let $R$ be the radius of the wheel and $q$ be the magnitude of the linear velocity of an arbitrary point on the outer circle of the wheel. If there were no gravity force the wet front in any moment $t > 0$ will be the circle centered at the origin and having radius $r(t)$ for which

$$r^2(t) = R^2 + (qt)^2.$$

Taking into account the gravity force, all these circles are falling with acceleration of gravity $g$. The equation of the 'falling circle' is

$$x^2 + \left(y + \frac{gt^2}{2}\right)^2 = r^2(t).$$

The envelope of the above family of circles is the boundary between the wet and the dry area in the plane. Let us determine it. Denote by $(X, Y)$ the coordinates of an arbitrary boundary point. Then

$$X^2 + \left(Y + \frac{gt^2}{2}\right)^2 = R^2 + q^2t^2.$$

Rewriting the above equation in the form

$$X^2 = -\frac{g^2t^4}{4} + (q^2 - gY)t^2 + R^2 - Y^2$$

we can consider it as a quadratic with respect to $t^2$. Since the point $(X, Y)$ belongs to the boundary, it is the ultimate among the 'wet' points with the same ordinate. Hence $X^2$ is the maximum of the quadratic function, i.e.

$$X^2 = R^2 + \frac{q^4}{g^2} - \frac{2q^2Y}{g}.$$

Thus

$$Y = -\frac{g}{2q^2}X^2 + \frac{gR^2}{2q^2} + \frac{q^2}{2g}$$

is the equation of the boundary between the wet and the dry area. In fact this parabola is the envelope of the one-parameter family of the parabolic trajectories of the water particles (see Fig. 7), parameterized with t but we obtain its equation as the envelope of the 'falling circles' (Fig. 8).

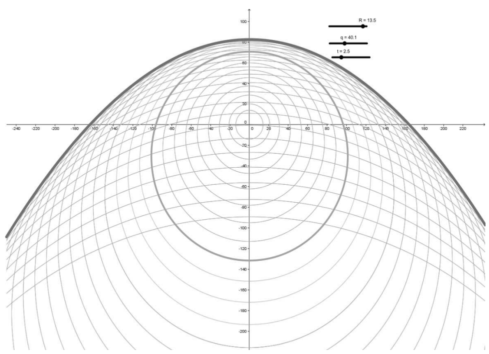

Fig. 8

*Dynamic drill.* Visualize the result from Problem 3 by DGS and explore the shape of the wet area with respect to the magnitude of $R$ and $q$.

## Final remarks

Our strong opinion is that the implementation of new technologies in teaching-learning process should go together with renovating the core of the curriculum. But such a renovation could be effective only if it follows evolutionary way, taking small steps and starting from a solid ground. From this perspective making errors is an instrument of the evolution and it is not a failure but additional resource for progress.

*Making sense to the theory.* The envelope of one-parameter family of smooth curves is rather familiar thing in university mathematics (e.g. the theory of ordinary differential equations). Sometimes the envelope is called *discriminant* of the family (remember the way we obtain the envelope in Problem 2).

Presenting envelopes in secondary school is a didactical challenge at least in two aspects: to find motivation for it and to find appropriate educational resources.

Highlighting the need of making sense to the theory, we believe that the kinematics applications are reasonable motivation to study envelopes. Visualizations by DGS are not only attractive but also rich in math content.

*Partition of the new knowledge.* The mathematics we use in the module does not go beyond the Bulgarian 10th grade extended curriculum which also includes the facts from kinematics we need for solving Problems 3 and 4. The problems that are included in the textbooks are limited to studying the movement of a stone thrown up vertically. Another problems deal with the decomposition of vectors, so our Problems 3 and 4 are just a link between these topics. However, Problems 3 and 4 are hard and depending on the classroom context they could be decomposed, for example, in the following way:

1st step: modeling the phenomenon.

2nd step: design of a dynamic construction of the model.

3rd step: examine the dynamic construction.

4th step: proof of the results.

Such decomposition allows assessing different student's activities and competences separately, forming in that way a kind of assessment spectrum.

*Taking chance on dynamic constructions.* In this module DGS is used mainly for illustrations but it is also a powerful tool for generating conjectures. The envelope $\kappa$ of $BC$ in the construction that follows Problem 1 coincides with circle in a special case shown in Fig. 1. Here the problem that $\kappa$ is always a circle appears clearly. Another interesting shapes (like this in Fig. 9) could be obtain by moving point $P$ in the DGS construction that gives Fig. 2. However, without deductive reasoning such kind of activities is nothig more than a mere play.

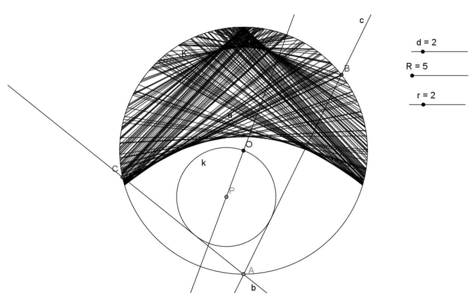


Fig. 9

*Warning!* Our recent classroom experience shows that any kind of exploration work on a dynamic construction during a lesson is extremely time consuming.

*Taking chance on mistakes.* This module was initially included in several teacher training courses in the period April–July 2010. In the first version of the module Parabola [2], only Problem 4 was under consideration. An error of the author who forgot to include the radius of the wheel in the calculations brought into the existence Problem 3: every time trying to visualize by DGS our (wrong) formulae instead something like Fig. 7, Fig. 5 appeared. The lesson to be learnt is that the inquiry based approach that includes DGS allows not only attractive teaching but also developing teaching materials.

Acknowledgment. The author thanks Albena Vassileva for pointing his error and for helping to design some DGS applets. The author thanks Pavel Boychev for the beautiful clips about the misty sprinkler and the wet wheel.

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