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COUSIN'S THEOREM AND TWO OTHER BASIC PROPERTIES EQUIVALENT TO IT

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Abstract. We present direct proofs of the equivalence between both the compactness and the connectedness of the interval $[a, b]$ and the Cousin's theorem in ways that allow their beauty to go through.

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It is known that the existence of δ-fine partition of any interval $[a, b]$ characterizes the least upper bound property (see e.g. in [1, Theorem 2.3.6]). Accordingly, as we shall show here, the existence of such partitions on $[a, b]$ also characterizes both the compactness and the connectedness of the interval. We shall demonstrate that such existence of partitions cannot only be applied to (delightfully) deduce either the compactness or the connectedness of the interval, but it can also be (delightfully) derived using only either one of these two properties. More than showing their equivalence per se, here we are particularly keen to present direct proofs, where no argument by contradiction in any part is used, in ways their beauty can go through.

Recall that the set X is compact if every open cover of X has a finite subcover. The set X is connected if it is not a union of two nonempty disjoint open subsets of X, or equivalently, the only nonempty subset of X that is open and closed in X is itself. Let δ be a *gauge* on $[a, b]$, that is a positive real-valued function on $[a, b]$. Let $\{I_i\}_{i=1}^n$ be a collection of nonoverlapping compact subintervals such that $\bigcup_{i=1}^n I_i = [a, b]$, and let $t_i \in I_i$ for each i. Then the set of ordered pairs $\{(I_i, t_i)\}_{i=1}^n$ is called a *δ-fine partition of* $[a, b]$ provided $I_i \subseteq [t_i - \delta(t_i), t_i + \delta(t_i)]$, for each $i \in \{1, \dots, n\}$. In such a case, every singleton $\{(I_i, t_i)\}$ is a δ-fine partition of I_i. If $x \in [a, b]$, then trivially $\{([x, x], x)\}$ is a δ-fine partition of $[x, x]$, as $[x, x] = \{x\} \subseteq [x - \delta(x), x + \delta(x)]$.

THEOREM 1. *The interval $[a, b]$ is compact if and only if, for any gauge δ, it has a δ-fine partition.*

Proof. ($\Rightarrow$) Let any gauge δ on $[a, b]$ be given. Let for $t \in [a, b]$,

$$I_\delta^t := (t - \delta(t), t + \delta(t)).$$

By the compactness of $[a,b]$, choose a finite open cover $\{I_\delta^{t_i}\}_{i=1}^n$ such that, after relabelling if necessary, we have $a \le t_1 < t_2 < \ldots < t_n \le b$, where n is chosen to be the minimum such that $\{I_\delta^{t_i}\}_{i=1}^n$ covers $[a,b]$. This implies that, for all $i,j \in \{1,\ldots,n\}$, we have

$$i \neq j \Longrightarrow I_\delta^{t_j} \nsubseteq I_\delta^{t_i} \quad \text{and} \quad I_\delta^{t_i} \nsubseteq I_\delta^{t_j}$$

and so

$$t_i < t_j \iff t_i + \delta(t_i) < t_j + \delta(t_j). \tag{1}$$

Now let $T := \{t_1,\ldots,t_n\}$, and

$$S := \{t \in T : [a,t] \text{ has a } \delta\text{-fine partition}\}.$$

Notice that $a \in I_\delta^{t_{i_0}}$ for some $t_{i_0} \in T$ by which $\{[a,t_{i_0}],t_{i_0}\}$ is a δ-fine partition of $[a,t_{i_0}]$, so that $t_{i_0} \in S$. Thus $S \neq \emptyset$. Now, if $t \in S$ and $b \notin I_\delta^t$, thus $t+\delta(t) \in [a,b]$, then there exists $t' \in T$ such that $t+\delta(t) \in I_\delta^{t'}$. It follows from (1) that $t < t'$, so that we can choose $x \in I_\delta^t \cap I_\delta^{t'}$ such that $t < x < t'$. Therefore, if $\mathcal{P}_t$ is a δ-fine partition of $[a,t]$, so is $\mathcal{P}_t \cup \{([t,x],t)\} \cup \{([x,t'],t')\}$ of $[a,t']$, thus $t' \in S$. We have just obtained (i) $S \neq \emptyset$; and (ii) for every $t_{i_k} \in S$, if $b \notin I_\delta^{t_{i_k}}$, there exists $t_{i_{k+1}} \in S$ such that $t_{i_k} < t_{i_{k+1}}$. Consequently, since T, thus S, is finite, we have an increasing sequence $t_{i_0} < t_{i_1} < \cdots$ in S, which eventually terminates at some $t_{i_p} \in S$ such that $b \in I_\delta^{t_{i_p}}$. Therefore, if $\mathcal{P}_{t_{i_p}}$ is a δ-fine partition of $[a,t_{i_p}]$, so is $\mathcal{P}_{t_{i_p}} \cup \{([t_{i_p},b],t_{i_p})\}$ of $[a,b]$.

($\Leftarrow$) Let $\mathcal{O} := \{U_\alpha\}_{\alpha\in\Lambda}$ be an open cover of $[a,b]$. Let for every $x \in [a,b]$, $\gamma(x)$ is a positive number such that $I_\gamma^x := (x-\gamma(x), x+\gamma(x)) \subseteq U_{\alpha_x}$, for some $U_{\alpha_x} \in \mathcal{O}$. Define $\delta(x) := \gamma(x)/2$, for each $x \in [a,b]$. Let $\{([x_{i-1},x_i],c_i)\}_{i=1}^n$, where $x_0 := a$ and $x_n := b$, be a δ-fine partition of $[a,b]$. Since $\{[c_i-\delta(c_i), c_i+\delta(c_i)]\}_{i=1}^n$ covers $[a,b]$, so does $\{I_\gamma^{c_i}\}_{i=1}^n$, and hence $\{U_{\alpha_{c_i}}\}_{i=1}^n$. This completes the proof. ∎

Theorem 2. *The interval $[a,b]$ is connected if and only if, for any gauge δ, it has a δ-fine partition.*

Proof. ($\Rightarrow$) Let any gauge δ on $[a,b]$ be given. Let

$$\mathcal{C}_\delta := \{[a,t] \subseteq [a,b] : [a,t] \text{ has a } \delta\text{-fine partition}\}.$$

Let $I := \bigcup_{J\in\mathcal{C}_\delta} J$. We assert that $I = [a,b]$. Since $[a,b]$ is connected, while $[a,b] \supseteq I \supseteq [a,a] \neq \emptyset$, it suffices to show that I is both open and closed in $[a,b]$. Let $x \in [a,t] \in \mathcal{C}_\delta$. If x is neither a nor an interior point of $[a,t]$, then $x = t$. If $t = b$, it follows that $[a,b] \in \mathcal{C}_\delta$, thus there is nothing to prove. Otherwise, if $\mathcal{P}_x$ is a δ-fine partition of $[a,x]$, so is $\mathcal{P}_x \cup \{([x,x+\eta_0/2],x)\}$ of $[a,x+\eta_0/2]$, where $\eta_0 := \min\{\delta(x), b-x\}$. Therefore $[a,x+\eta_0/2] \in \mathcal{C}_\delta$, and so x is an interior point. This shows that I is open. Now, let $x \in [a,b]$ be a limit point of I. Then either x is contained in some $J \in \mathcal{C}_\delta$, hence in I, or there exists $[a,x_1] \in \mathcal{C}_\delta$ such that $x_1 \in (x-\delta(x),x)$. In the latter case, if $\mathcal{P}_{x_1}$ is a δ-fine partition of $[a,x_1]$, so is

$\mathcal{P}_{x_1} \cup \{([x_1, x], x)\}$ of $[a, x]$, and thus $[a, x] \in \mathcal{C}_\delta$, so that $x \in I$. This proves I is closed, thus the assertion $I = [a, b]$. The fact that $b \in I$ then implies that $[a, b] \in \mathcal{C}_\delta$.

($\Leftarrow$) Let $\emptyset \neq A \subseteq [a, b]$, A is both open and closed in $[a, b]$. We wish to show that $A = [a, b]$. First notice, since A is also closed in $[a, b+1]$, $B := [a, b+1] \setminus A$ is nonempty and open in $[a, b+1]$. Let for every $x \in [a, b+1]$, $\gamma(x)$ is a positive number, and

$$I_\gamma^x := \begin{cases} [a, a+\gamma(x)), & \text{if } x = a, \\ (b-\gamma(x), b], & \text{if } x = b, \\ (x-\gamma(x), x+\gamma(x)), & \text{otherwise.} \end{cases}$$

Now, for each $x \in [a, b+1]$, choose $\gamma(x)$ such that $I_\gamma^x \subseteq A$ if $x \in A$, and $I_\gamma^x \subseteq B$ if $x \in B$. Define $\delta(x) := \gamma(x)/2$, for each $x \in [a, b]$. Thus δ is a gauge on $[a, b]$. Notice that, by the choice of $\gamma(x)$, for every $x \in [a, b]$, we have

$$A \cap I_\gamma^x \neq \emptyset \implies I_\gamma^x \subseteq A. \tag{2}$$

Let $\{([x_{i-1}, x_i], c_i)\}_{i=1}^n$, where $x_0 := a$ and $x_n := b$, be a δ-fine partition of $[a, b]$. If $x \in A$, then $x \in [x_{i-1}, x_i] \subseteq [c_i - \delta(c_i), c_i + \delta(c_i)] \cap [a, b] \subseteq I_\gamma^{c_i}$, for some $i \in \{1, \ldots, n\}$. Thus by (2), $c_i \in I_\gamma^{c_i} \subseteq A$. Therefore $A \subseteq \cup_{c_i \in A}[x_{i-1}, x_i]$. Conversely, if $c_i \in A$, since $c_i \in I_\gamma^{c_i}$, it follows that $[x_{i-1}, x_i] \subseteq [c_i - \delta(c_i), c_i + \delta(c_i)] \cap [a, b] \subseteq A$. Thus $\cup_{c_i \in A}[x_{i-1}, x_i] \subseteq A$. We obtain $A = \bigcup_{c_i \in A}[x_{i-1}, x_i]$. Next, for each i such that $c_i \in A$, choose and label $[x_{i-1}, x_i]$ as $[a_i, b_i]$ such that, after labeling, $\{[a_i, b_i]\}_{i=1}^l$ covers $[a, b]$ minimally and $a_1 < a_2 < \cdots < a_l$ and $b_1 < b_2 < \cdots < b_l$. Since A is open in $[a, b]$, if $x \in A$, then either $x \in \{a, b\}$ or x is an interior point of A. Since a_1 is not b nor an interior point, $a_1 = a$. Similarly, $b_l = b$. Now let

$$T := \{b_i \in \{b_1, \ldots, b_l\} : [a, b_i] \subseteq A\}.$$

Since $[a, b_1] \subseteq A$, $T \neq \emptyset$. Let $b_i \in T$ and $b_i \neq b$. Then b_i is an interior point, so that $b_i \in [a_j, b_j]$, for some j with $b_i < b_j$. Therefore $[a, b_j] = [a, b_i] \cup [a_j, b_j] \subseteq A$, and so $b_j \in T$. Since T is finite, it follows that there exists an increasing sequence $b_{i_1} < b_{i_2} < \cdots$ in T that eventually terminates at some $b_{i_k} = b$, meaning that $[a, b] \subseteq A$, and so $A = [a, b]$. This completes the proof. ■

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