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A SHORT ELEMENTARY PROOF OF THE INFINITUDE OF PRIME NUMBERS

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Abstract. We present a new short proof of the infinitude of prime numbers. This is a proof by contradiction, and it is based on the prime factorization of a positive integer.

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A prime number (or briefly, a prime) is an integer greater than 1 that is divisible only by 1 and itself. Mathematicians have been studying prime numbers, their properties and related results for over twenty-three centuries. Ancient Greek mathematicians knew that there are infinitely many primes. Namely, circa 300 B.C. Euclid of Alexandria, from the *Pythagorean School*, proved in his book *Elements* the following celebrated result as rendered into modern language from the Greek (see [1, Book IX, Proposition 20, p. 412] and [4]):

If a number be the least that is measured by prime numbers, it will not be measured by any other prime number except those originally measuring it.

The above assertions is equivalent to the following famous Euclid's theorem on the infinitude of prime numbers.

THEOREM (Euclid) [3, p. 3]. *There are infinitely many prime numbers*

Nowadays, one can find numerous proofs of the infinitude of prime numbers. In 2018, the author of this article provided in [2] a comprehensive historical survey of 183 different proofs of this famous theorem. Furthermore, numerous elementary proofs of the infinitude of prime numbers in different arithmetic progressions were listed.

Here we present a simple elementary proof of the infinitude of prime numbers.

Proof of Theorem. Suppose that $p_1 = 2 < p_2 = 3 < \cdots < p_k$ are all prime numbers. Take $n = p_1 p_2 \cdots p_k$. Then $n-1$ can be factored as $n-1 = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ for some k-tuple $(e_1, e_2, \dots, e_k)$ of nonnegative integers $e_1, e_2, \dots, e_k$. Then taking $s = \max\{e_1, e_2, \dots, e_k\}$, we find that

$$n-1 = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k} = \frac{p_1^s p_2^s \cdots p_k^s}{p_1^{s-e_1} p_2^{s-e_2} \cdots p_k^{s-e_k}} = \frac{n^s}{a},$$

where $n = p_1 p_2 \cdots p_k$, $a = p_1^{s-e_1} p_2^{s-e_2} \cdots p_k^{s-e_k}$ and s are positive integers. From the above inequality it follows that

$$a = \frac{n^s}{n-1} = \frac{(n^s - 1) + 1}{n-1} = \sum_{i=0}^{s-1} n^i + \frac{1}{n-1},$$

whence it follows that $1/(n-1) = a - \sum_{i=0}^{s-1} n^i$ is a positive integer. This shows that $n = 2$, i.e. $2 = p_1 p_2 \cdots p_k$. This equality implies that $k = 1$, i.e. that $p_1 = 2$ is the greatest prime number. This contradiction completes the proof. ■

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