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THE SHARKOVSKII THEOREM AS A TOOL FOR INCREASING THE LEVEL OF COMBINATORIAL THINKING

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Abstract. The Sharkovskii theorem is an important mathematical result of the last century, which significantly helped the development of a modern mathematical field – chaos theory. The Sharkovskii theorem is related to an interval on the real axis and a single-valued function. However, it was later generalized in various ways. This article will focus on the multivalued generalization of the Sharkovskii theorem, which has also found application in the theory of differential equations. We will show that the considered multivalued generalization of the Sharkovskii theorem is also suitable in mathematics education, especially for increasing the level of combinatorial thinking. We present some examples for pupils of different levels.

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1. Introduction

The butterfly effect is one of the most famous symbols of chaos theory. We recall that the butterfly effect means a very sensitive dependence on initial conditions. Illustratively, fluttering butterfly wings at one end of the world can cause significant changes in weather at the other.

The theory of chaos began to emerge as early as the end of the 19th century (Henri Poincaré). Still, its fundamental development took place in the second half of the twentieth century. Chaos theory has found application in physics, biology, meteorology, and sociology. It is a counterweight to the specialization of some disciplines and leads to a more comprehensive view of the issue. We recommend the book by Gleick [8] for more exciting information about chaos theory.

For example, biologist Robert May and meteorologist Edward Lorenz dealt with chaos theory. Based on their research, James Yorke and his student Tien-Yien Li [12] published a paper in which they mathematically described chaos in a one-dimensional case. Specifically, they showed that a cycle of period 3 implies the cycles of all periods – the result of J. Yorke and T.-Y. Li is only a particular case of a more general result obtained by A. N. Sharkovskii [14]. However, A. N. Sharkovskii did not associate his purely mathematical result with natural phenomena, leading to chaos theory.

Let us explain the notion of the period, the Sharkovskii ordering, and the Sharkovskii theorem.

Definition 1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function. We say that $a \in \mathbb{R}$ is a periodic point of period n of f if $f^n(a) = a$ and $f^j(a) \neq a$ for $0 < j < n$.

The Sharkovskii ordering for all positive integers was introduced in [14]. In this ordering, the largest number is 3, and the smallest number is 1:

$$3 \rhd 5 \rhd 7 \rhd 9 \rhd \cdots \rhd 2\cdot 3 \rhd 2\cdot 5 \rhd 2\cdot 7 \rhd 2\cdot 9 \rhd \cdots \rhd 2^2\cdot 3 \rhd 2^2\cdot 5 \rhd 2^2\cdot 7 \rhd 2^2\cdot 9$$
$$\rhd \cdots \rhd 2^n\cdot 3 \rhd 2^n\cdot 5 \rhd 2^n\cdot 7 \rhd 2^n\cdot 9 \rhd \cdots \rhd 2^{n+1}\cdot 3 \rhd 2^{n+1}\cdot 5 \rhd 2^{n+1}\cdot 7 \rhd 2^{n+1}\cdot 9$$
$$\rhd \cdots \rhd 2^{n+1} \rhd 2^n \rhd \cdots \rhd 2^2 \rhd 2 \rhd 1.$$

Theorem 1. (Sharkovskii, [14]). *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function. If f has a periodic point of period n with $n \rhd k$, then f also has a periodic point of period k.*

The Sharkovskii theorem can be used to describe the motion of oscillators as a pendulum [6] or a double pendulum [11]. Let us note that it is possible to compare the future of the pandemic to the movement of a double pendulum [9].

It was explained, e.g., in [1], that the previous Sharkovskii theorem cannot be used to detect the periodic solutions of differential equations in contrast to the multivalued version of the Sharkovskii theorem, which was introduced in [3], see also [2, 4] for subsequent generalization and the specification of exceptions.

Let us remark that the ability to find periodic solutions of differential equations can help understand oscillators' motion better; for more details, see, e.g., [7].

In Section 2, we show that the considered multivalued generalization of the Sharkovskii theorem is also suitable in mathematics education, especially for increasing the level of combinatorial thinking for pupils. It seems that the multivalued version of the Sharkovskii theorem is a more convenient tool in mathematics education than the classical Sharkovskii theorem (for single-valued function) or some other generalizations of the classical Sharkovskii theorem, see for example [5], [10] or [13].

The multivalued version of the Sharkovskii theorem was stated in terms of M-maps and n-orbits.

Definition 2. We say that a multivalued mapping φ from $\mathbb{R}$ to $\mathbb{R}$, we write $\varphi : \mathbb{R} \rightsquigarrow \mathbb{R}$, is an M-map if the following conditions are satisfied:

- the mapping φ is upper-semicontinuous, which means that for every open interval (a, b), it holds that the set $\varphi^{-1}(a, b) = \{\, x \in \mathbb{R} : \exists y \in (a, b),\ y \in \varphi(x) \,\}$ is an open interval too;
- for every $x \in \mathbb{R}$, it holds that the set $\varphi(x)$ is either a closed and bounded interval or a single point.

Remark 1. Notice that every single-valued continuous function is upper semicontinuous. Of course, it is unnecessary (and probably not even appropriate) to explain to pupils the notion of upper-semicontinuous function.

DEFINITION 3. Let $\varphi\colon \mathbb{R} \rightsquigarrow \mathbb{R}$ be an M-map. By an n-orbit of φ, we mean a sequence $\{x_1, x_2, \ldots, x_n\}$ of points $x_j \in \mathbb{R}$ such that $x_{j+1} \in \varphi(x_j)$, $j = 1, 2, \ldots, n$, where $x_{n+1} = x_1$ and the previous sequence cannot be formed by going p-times around a shorter m-orbit, where $m \cdot p = n$.

THEOREM 2. [3] *Let $\varphi\colon \mathbb{R} \rightsquigarrow \mathbb{R}$ be an M-map and let φ have an n-orbit, where $n = 2^m \cdot q$, $m \in \mathbb{N}_0$, and q is odd, and let n be maximal in the Sharkovskii ordering such that there exists an n-orbit.*

1. *If $q > 3$, then φ has a k-orbit for every k, $n \rhd k$, possibly except $k = 2^{m+2}$.*
2. *If $q = 3$, then φ has a k-orbit for every k, $n \rhd k$, possibly except $k = 2^{m+1} \cdot 3$, 2^{m+2}, 2^{m+1}.*
3. *If $q = 1$, then φ has a k-orbit for every k, $n \rhd k$.*

2. Sharkovskii theorem in mathematics education

Now, we want to show some examples that may be useful in mathematics education. The first example can help to increase numeracy, especially the comparing of numerical magnitudes and the factorization of an integer.

EXAMPLE 1. Compare the following numbers in the Sharkovskii ordering:

a) 47, 56;

b) 9, 16;

c) 2021, 2019.

Solution of Example 1. In the Sharkovskii ordering of all positive integers, the sequence of all odd numbers (except 1) comes first, i.e.

$$3 \rhd 5 \rhd 7 \rhd 9 \rhd \cdots$$

Then follows the sequence of numbers formed as the product of the number 2 and an odd number, i.e.,

$$2 \cdot 3 \rhd 2 \cdot 5 \rhd 2 \cdot 7 \rhd 2 \cdot 9 \rhd \cdots$$

Then follows the sequence of numbers formed as the product of the number 2^2 and an odd number, i.e.,

$$2^2 \cdot 3 \rhd 2^2 \cdot 5 \rhd 2^2 \cdot 7 \rhd 2^2 \cdot 9 \rhd \cdots$$

In general, the sequence formed as the product of the number 2^n and an odd number, i.e.,

$$2^n \cdot 3 \rhd 2^n \cdot 5 \rhd 2^n \cdot 7 \rhd 2^n \cdot 9 \rhd \cdots,$$

is followed by the sequence formed as the product of the number 2^{n+1} and an odd number, i.e.,

$$2^{n+1} \cdot 3 \rhd 2^{n+1} \cdot 5 \rhd 2^{n+1} \cdot 7 \rhd 2^{n+1} \cdot 9 \rhd \cdots$$

At the end of the Sharkovskii ordering of all positive integers is the sequence formed by powers of 2, i.e.,

$$\cdots \rhd 2^{n+1} \rhd 2^n \rhd \cdots \rhd 2^2 \rhd 2 \rhd 1.$$

Thus, when solving Example 1, it is enough to decompose the considered numbers into the product of a power of the number 2 and an odd number and then consider the order of the previous sequences.

a) $47 \rhd 56 = 2^3 \cdot 7$;

b) $9 \rhd 16 = 2^4$;

c) $2019 \rhd 2021$.

The following two combinatorial examples also seem helpful for understanding the concept of function and graph reading.

Example 2. Let us consider the following map $\varphi : \mathbb{R} \rightsquigarrow \mathbb{R}$ (see Figure 1):

- $\varphi(x) = \{2\}$, for $x \in (-\infty, 0]$,
- $\varphi(x) = \{x + 2\}$, for $x \in [0, 1)$,
- $\varphi(x) = \{3\}$, for $x \in [1, 2)$,
- $\varphi(x) = [0, 4]$, for $x = 2$,
- $\varphi(x) = \{3\}$, for $x \in (2, 4)$,
- $\varphi(x) = [0, 3]$, for $x = 4$,
- $\varphi(x) = \{1\}$, for $x \in (4, +\infty)$.

Find a 2-orbit, a 3-orbit, a 4-orbit, a 5-orbit, a 6-orbit and a 12-orbit of φ.

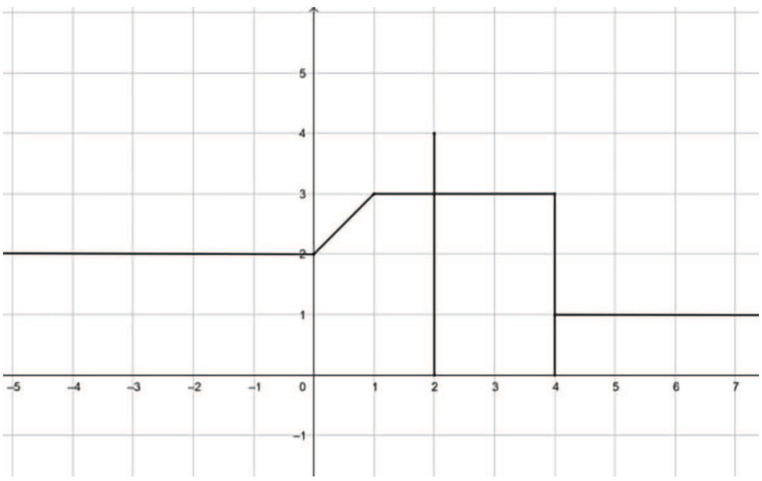

Figure 1. Function from Example 2

Solution of Example 2. It follows from Definition 3 that for a 2-orbit, we must find the points a and b such that $a \in \varphi(b)$ and $b \in \varphi(a)$. It is easy to see from the graph of the map φ that there exists a 2-orbit $\{2, 4\}$. We can also see another 2-orbit $\{0, 2\}$.

For a 3-orbit, we must find the points a, b, and c with the properties $b \in \varphi(a)$, $c \in \varphi(b)$ and $a \in \varphi(c)$. We can see from the graph that there exists a 3-orbit $\{0, 2, 4\}$.

Then, since $2 \in \varphi(2)$ (we can say that 2 is a fixed point of φ), it suffices to include the next point 2 to the previous 3-orbit and in this way to obtain a 4-orbit $\{0, 2, 2, 4\}$.

Knowledge of the previous orbits then gives a 5-orbit $\{2, 0, 2, 4, 0\}$, a 6-orbit $\{0, 2, 0, 2, 4, 2\}$, and a 12-orbit $\{0, 2, 4, 2, 2, 2, 2, 4, 0, 2, 2, 2\}$. △

The following map was presented in [3] to illustrate that the exceptions mentioned in the multivalued Sharkovskii theorem can occur.

Example 3. Let us consider the following M-map $\varphi\colon \mathbb{R} \rightsquigarrow \mathbb{R}$ (see Figure 2):

- $\varphi(x) = \{2\}$, for $x \in (-\infty, 0]$,
- $\varphi(x) = \{x + 2\}$, for $x \in [0, 1)$,
- $\varphi(x) = \{3\}$, for $x \in [1, 2)$,
- $\varphi(x) = [3, 4]$, for $x = 2$,
- $\varphi(x) = \{3\}$, for $x \in (2, 4)$,
- $\varphi(x) = [0, 3]$, for $x = 4$,
- $\varphi(x) = \{1\}$, for $x \in (4, +\infty)$.

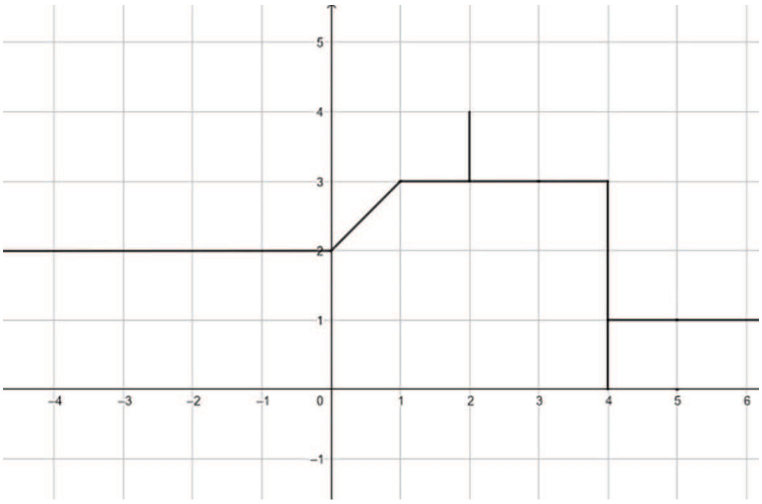

Figure 2. Function from Example 3

Knowing that two of the orbits of φ among the following eight do not exist, identify the remaining six: 1-orbit, 2-orbit, 3-orbit, 4-orbit, 5-orbit, 6-orbit, 8-orbit, 2025-orbit.

Solution of Example 3. It is enough to find six orbits of the previous 8.

In Figure 2, let us imagine the graph of the function $f(x) = x$. The intersection of the graphs φ and f is only the point $(3, 3)$, which means that the mapping φ has only one 1-orbit, namely $\{3\}$.

Using the same ideas as in Example 2, we can obtain that the mapping φ has a 2-orbit $\{2,4\}$, a 3-orbit $\{0,2,4\}$, a 5-orbit $\{0,2,4,2,4\}$ and a 2025-orbit $\{0,2,4,\dots,2,4\}$, where the subsequence $\{2,4\}$ repeats 1012 times.

Now, we can also conclude that the considered M-map does not have 4-orbit and 6-orbit. △

Of course, we adapt the examples mentioned above to the pupils' knowledge and age to avoid unnecessarily burdening them with superfluous concepts. We will focus on the type of Example 1 and offer their variants for pupils of different levels.

Introducing the complete Sharkovskii ordering is unnecessary, especially for younger pupils. We can limit ourselves to its part and choose another name (for example, a magic ordering). Let us offer the variant of Example 1 for pupils of primary education, lower secondary education, and upper secondary education, respectively. We can obtain the correct solution similar to the solution of Example 1.

- Primary education.
  Compare the following numbers in the Sharkovskii ordering: 9, 13.
  (Correct answer $9 \rhd 13$).
- Lower secondary education.
  Compare the following numbers in the Sharkovskii ordering: 17, 7, 21, 15.
  (Correct answer $7 \rhd 15 \rhd 17 \rhd 21$).
- Upper secondary education.
  Compare the following numbers in the Sharkovskii ordering: 11, 64, 78, 20.
  (Correct answer $11 \rhd 78 = 2 \cdot 39 \rhd 20 = 2^2 \cdot 5 \rhd 64 = 2^6$).

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