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# CUBIC EQUATIONS AND GEOMETRIC CONSTRUCTIONS

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**Abstract.** Examples are given of parametric families of equations of the third degree, for which all roots are expressed by square radicals. The problem of constructing a quadrilateral inscribed in a given semicircle by ruler and compass alone is discussed. It is shown that the problem of constructing an isosceles triangle if its three bisectors are given is equivalent to the problem of trisecting an angle. A connection was established between the problem of trisection of an angle and the problem of constructing a regular polygon.

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## 1. Introduction

Three problems of antiquity stimulated the development of mathematics for many centuries – squaring the circle, trisecting the angle and doubling the cube [1]. The problem of doubling the cube is reduced to constructing, using ruler and compasses alone, the real root of the cubic equation $z^3 = 2$. The angle trisection problem leads to the problem of constructing a root of the equation $4z^3 - 3z - \cos\theta = 0$. In the 19th century it was proven that these and some other problems cannot be solved using compass and ruler alone.

Let us recall the basic principles of geometric constructions using a compass and ruler. The operations of addition, subtraction, multiplication, division, and extraction of the square root (arithmetic of a non-negative number) are called basic operations. In addition, it is assumed that a unit segment is given (or chosen). It turns out that the following theorem is true.

THEOREM. *In order to construct a segment with a compass and a ruler, it is necessary and sufficient that the length of the desired segment can be expressed through the lengths of the given segments using a finite number of basic operations.*

It has been proven that in the general case the roots of a cubic equation $z^3+az^2+bz+c = 0$ are not expressed using the basic operations with real coefficients $a$, $b$ and $c$, and therefore cannot be constructed with a compass and ruler [1]. In this case we will say that the cubic equation is not solvable in square radicals. If the equation is solvable in square radicals, then its positive roots can be constructed with a compass and ruler [1].

Next, we will show that there are infinitely many cubic equations solvable in square radicals. In particular, we will give non-trivial examples where some classical problems on constructions with a compass and a ruler are solvable. In addition, a connection will be established between problems on constructing an isosceles triangle by its bisectors and a regular polygon with the problem of trisection of an angle. The purpose of this article is to convince the reader that the main drawback of Cardano's formula (the presence of a cubic root in the formula) can sometimes be overcome.

## 2. Cubic equations

The Cardano's formula (Ferro-Tartaglia-Euler formula) for the roots of the equation

$$x^3 + px + q = 0 \tag{1}$$

looks like this [3]

$$x = \sqrt[3]{-\frac{q}{2} + \sqrt{D}} + \sqrt[3]{-\frac{q}{2} - \sqrt{D}}, \qquad D = \frac{p^3}{27} + \frac{q^2}{4},$$

and turns out to be useless for constructing positive roots with a compass and ruler. However, if $p = -3mn$, $q = -(m^3 + n^3)$, then for integers $m$ and $n$ of the same parity (both even or both odd) all roots in this formula are integers:

$$\sqrt{D} = \frac{|m^3 - n^3|}{2}, \qquad \sqrt[3]{-\frac{q}{2} \pm \sqrt{D}} = \frac{m^3 + n^3}{2} \pm \frac{|m^3 - n^3|}{2}.$$

And the root $x_1 = m + n$ of the equation $x^3 - 3mnx - (m^3 + n^3) = 0$ is also an integer. This infinite family of equations is the solution to Exercise 4.1 [3, page 74].

The quantity $D = p^3/27 + q^2/4$ is called the discriminant of equation (1). It is known that if $D > 0$, then equation (1) with real coefficients $p$ and $q$ has one real root [3]. If $D < 0$ (the so-called "irreducible case"), then equation (1) has three real roots.

If $D = 0$, then equation (1) has a double root. Using Cardano's formula, we obtain the root $x = 2\sqrt[3]{-q/2}$. However, this root can be expressed without extracting the cube root. Using the equality $q^2/4 = -p^3/27$, we obtain a rational expression through the coefficients $p$ and $q$ for this root as $x = 2\sqrt[3]{-q/2} = -q/\sqrt[3]{q^2/4} = -q/\sqrt[3]{-p^3/27} = 3q/p$. In this case, the expansion $x^3 + px + q = (x + 3q/(2p))^2(x - 3q/p)$ is valid and all positive roots can be constructed with a compass and ruler.

Thus, in special cases, when the coefficients of the original equation are related by some additional relations, it is sometimes possible to express the roots of the equation through the coefficients using basic operations, without using cubic roots and Cardano formulas. E.g., if in equation (1) the coefficients $p$ and $q$ are related by the relation

$$q = -2(p + 4), \tag{2}$$

then $x = 2$ is the root of equation (1), and the other two roots are expressed in square radicals. For example, the roots of the equation $x^3 - 19x + 30 = 0$ are integers 2, 3 and $-5$.

Let us give two more factorizations of a polynomial $x^3 + px + q$ into linear factors, when its positive roots can be constructed with a compass and a ruler. If the equality $2p^3 + 27q^2 = 0$ holds for the coefficients $p$ and $q$ of equation (1), then

$$x^3 + px + q = \Big(x + \frac{3q}{p}\Big)\Big(x - \frac{3q}{2p}(1+\sqrt{3})\Big)\Big(x - \frac{3q}{2p}(1-\sqrt{3})\Big).$$

If the equality $q^2 = -(p/3)^3(2 \pm \sqrt{2})$ holds for the coefficients $p$ and $q$, then

$$x^3 + px + q = \Big(x - \frac{3p^2q}{p^3 + 27q^2}\Big)(x - x_2)(x - x_3),$$

where $x_{2,3} = \pm\dfrac{6p^2q}{2p^3(\sqrt{3} \mp 2) + 27q^2(\sqrt{3} \mp 1)}$. Readers will easily be convinced of the validity of these expansions on their own.

### 3. Newton quadrilaterals

In his book [5], Newton devoted 16 pages to analysis and various ways of deriving the equation

$$x^3 - (a^2 + b^2 + c^2)x - 2abc = 0, \tag{3}$$

where $a$, $b$, $c$ and $x$ are the sides of a quadrilateral inscribed in a semicircle (here $x$ is the diameter of the circle). Equation (3) is cubic with respect to $x$ and quadratic with respect to $a$, $b$, $c$. Obviously, if the diameter and two of the three sides of a quadrilateral are given, then the fourth side is easily constructed with compass and ruler. And from equation (3) it is clear that any of the sides $a$, $b$, $c$ of the quadrilateral is expressed in square radicals in terms of the other three. However, in the general case, given lengths $a$, $b$ and $c$ of the three sides, it is impossible to construct the fourth side of the quadrilateral, that is, the diameter of the circle [4].

Let us analyze equation (3). Its discriminant satisfies the inequality

$$D = \frac{\big(-(a^2+b^2+c^2)\big)^3}{27} + \frac{(-2abc)^2}{4} = \frac{-(a^2+b^2+c^2)^3 + 27a^2b^2c^2}{27} \leqslant 0,$$

which follows from the inequality $(a^2+b^2+c^2)/3 \geq \sqrt[3]{a^2b^2c^2}$ of the arithmetic mean and geometric mean of three non-negative numbers $a^2$, $b^2$, $c^2$. Therefore, equation (3) has three real roots $x_1$, $x_2$, $x_3$. In addition, according to Vieta's theorem, for equation (3) we have the relations $x_1 + x_2 + x_3 = 0$ and $x_1x_2x_3 = -(-2abc) = 2abc > 0$. It follows that for any three positive $a$, $b$ and $c$, only one root of equation (3) is positive.

Let us consider a special case. Let the coefficients $p = -(a^2 + b^2 + c^2)$ and $q = -2abc$ of equation (3) satisfy condition (2). For example, if $a = b = \sqrt{2 - \sqrt{3}}$, $c = \sqrt{3}$, we get the equation $x^3 - (7 - 2\sqrt{3})x - 2(2\sqrt{3} - 3) = 0$, which has a root $x = 2$. That is, this equation with irrational coefficients has an integer root that can be constructed with a compass and ruler.

Let us present two more series of equations (3) solvable in square radicals. If $a^2 + b^2 + c^2 = 4a^2b^2c^2 - 1$ for complex numbers $a$, $b$ and $c$, equation (3) is solvable in square radicals because it has a root $2abc$. And if $a^2(2a^2 + b^2 + c^2) = 4b^2c^2$, equation (3) is solvable by square radicals because it has a root $2bc/a$.

## 4. Constructing a triangle by using its bisectors

It is known that for any given three positive numbers $l_a$, $l_b$ and $l_c$ there is a unique triangle having $l_a$, $l_b$ and $l_c$ as its bisectors. However, in the general case its construction by compass and ruler is impossible. Moreover, even constructing the isosceles triangle by compass and ruler if its three bisectors are given is generally impossible. Consider the equation [2]

$$y^3 - 2ty^2 - 3y/4 + t = 0, \tag{4}$$

where $t = l_c/(2l_a)$, $y = \sin(A/2)$, and $l_a = l_b$, $l_c$ are the lengths of the bisectors, and $A = B$ are the angles of an isosceles triangle.

Having solved equation (4), we find the sides by using the formula for the bisector:

$$l_a = \frac{2bc}{b+c}\cos\frac{A}{2} = \frac{2ac}{a+c}\cos\frac{A}{2} = \frac{2a \cdot 2a\cos A}{a + 2a\cos A}\cos\frac{A}{2} = \frac{4a\cos A}{1 + 2\cos A}\cos\frac{A}{2}.$$

Expressing $\cos A$ and $\cos(A/2)$ in terms of $y = \sin(A/2)$, we obtain an expression for the side $a$ and similarly for the side $c = l_c \cdot 2\operatorname{ctg} A$:

$$a = l_a\,\frac{3 - 4y^2}{4(1 - 2y^2)\sqrt{1 - y^2}}, \qquad c = l_c\,\frac{1 - 2y^2}{y\sqrt{1 - y^2}}. \tag{5}$$

Therefore, if $y = \sin(A/2)$ can be expressed in square radicals, the sides $a = b$ and $c$ can also be expressed in square radicals. Then the isosceles triangle can be constructed if its bisectors are given. The table on the next page shows some special cases when such construction is possible by compass and ruler. The last line of the table gives approximate values of $a/l_a = b/l_b$ and $c/l_c$ because the exact expressions in terms of square radicals are too cumbersome. For comparison, note that for the equilateral triangle $a/l_a = b/l_b = c/l_c = 2\sqrt{3} \approx 1{,}15$.

| $l_c/(2l_a)$ | $\sin(A/2)$ | $A = B$ | $C$ | $a/l_a = b/l_b$ | $c/l_c$ |
|---|---|---|---|---|---|
| $\dfrac{1}{4}$ | $\dfrac{\sqrt{5}-1}{4}$ | $36°$ | $108°$ | $\dfrac{\sqrt{5}+1}{\sqrt{10+2\sqrt{5}}}$ | $\dfrac{6+2\sqrt{5}}{\sqrt{10+2\sqrt{5}}}$ |
| $11/56$ | $1/4$ | $\approx 83{,}6°$ | $\approx 12{,}8°$ | $33\sqrt{5}/20$ | $\sqrt{5}/10$ |
| $23/84$ | $1/3$ | $\approx 38{,}9°$ | $\approx 102{,}2°$ | $69\sqrt{2}/112$ | $7\sqrt{2}/4$ |
| $11/6$ | $2/3$ | $\approx 28{,}96°$ | $\approx 122{,}1°$ | $22\sqrt{15}/105$ | $14\sqrt{15}/15$ |
| $\sqrt{27/2}/4$ | $\sqrt{6}/4$ | $\approx 75{,}5°$ | $\approx 29°$ | $3\sqrt{10}/5$ | $2/\sqrt{15}$ |
| $\sqrt{27/2}/18$ | $(\sqrt{6}-\sqrt{2})/4$ | $30°$ | $120°$ | $\sqrt{6}/3$ | $2\sqrt{3}$ |
| $\approx 0{,}2826$ | $171/500$ | $\approx 40{,}0°$ | $\approx 100{,}0°$ | $\approx 0{,}879$ | $\approx 2{,}38$ |
| $\sqrt{27/2}/8$ | $(\sqrt{102}-\sqrt{6})/16$ | $\approx 57{,}1°$ | $\approx 65{,}8°$ | $\approx 1{,}09$ | $\approx 1{,}29$ |

## 5. Trisecting an angle

From the formula $\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$ for the cosine of a triple angle, we obtain

$$4\cos^3\frac{A}{2} - 3\cos\frac{A}{2} - \cos\frac{3A}{2} = 0. \tag{6}$$

That is, for a given angle $3A/2$, the cosine of the angle $A/2$ is found from the cubic equation (6). Consequently, if equation (6) can be solved by square radicals, trisecting the angle $3A/2$ by compass and ruler is possible.

The formula $\sin^2\alpha + \cos^2\alpha = 1$ and the above results for $A = 36°$ give $\sin(A/2) = (\sqrt{5}-1)/4$. Therefore, $\cos(A/2) = \sqrt{(5+\sqrt{5})/8}$ is a root of equation (6), in which $\cos(3A/2) = (\sqrt{5}-1)\sqrt{(5+\sqrt{5})/2}\,\Big/\,4$. Therefore, the angle $3A/2 = 54°$ can be divided by compass and ruler into three equal parts.

It is known that trisecting of the angle $3A/2 = 60°$ is impossible [1]. However, it is possible to approximate $\sin 20°$ with any accuracy by a rational number, for example, taking $\sin 20° \approx 171/500$. Then $\approx \cos 20°$ is a root of equation (6), where $\cos(3A/2) = 33259\sqrt{220759}/31250000 \approx 0{,}50006$. Therefore, the angle $3A/2 \approx 59{,}996°$ can be divided by compass and ruler into three equal parts. In this way we obtain an approximate solution to the problem of trisecting the angle $60°$.

Now note that if $y = \sin(A/2)$ can be expressed in square radicals, $\cos(A/2)$ and $\cos(3A/2)$ can also be expressed in square radicals. Therefore, if one can construct an arbitrary isosceles triangle by compass and ruler using its three bisectors $l_a = l_b$, $l_c$, then one can trisect the angle $3A/2$. And vice versa, if it is possible to trisect an angle $0 < 3A/2 < 135°$, then $\cos(A/2)$ and $y = \sin(3A/2)$ can be expressed in square radicals. Then we find the bisectors $l_a = l_b$ and $l_c$, which are determined by (4). And finally, we construct the triangle using expressions (5). In this sense, these two problems are equivalent. And the problem of constructing a triangle using its three arbitrary bisectors turn out to be more difficult than the problem of trisecting an angle.

## 6. Regular polygons

The problem of trisecting an angle is also related to the problem of constructing a regular polygon [1]. According to the Gauss-Wanzel theorem, the regular $n$-polygon can be constructed with compass and ruler if $n = 2^k p_1 p_2 \cdots p_m$, where $k$ is a natural number or 0, $p_i$ are different Fermat prime numbers $(3, 5, 17, 257, \dots)$. Therefore, for such numbers $n$ it is possible to construct $\sin(180°/n) = a/(2R)$, where $a$ is the side of the regular $n$-polygon which is inscribed in a circle of radius $R$. Then we construct $\cos(180°/n)$ and, using (6), construct $\cos(3(180°/n))$. That is, the problem of trisecting the angle $3(180°/n)$ is solved. For example, trisecting the angles $3(180°/n)$, where $n = 5,\ 10,\ 15,\ 17,\ 34,\ 51,\ \dots$ by compass and ruler alone is possible.

In conclusion, note that cubic equations often arise in physical and technical problems. For example, the van der Waals equation (the equation of state of a real gas) is a cubic equation with respect to the density of the gas, its volume and mass.

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