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APPLICATION OF DIFFERENT METHODS FOR SOLVING LINEAR DIFFERENCE EQUATIONS WITH VARIABLE COEFFICIENTS

Mehmed Nurkanović and Mirsad Trumić

Abstract. Several methods for solving linear difference equations are considered: the operator method, the method of invariants, the method of generating functions, the Z-transform method, and the factorial series method. The application of these methods is demonstrated by solving higher-order linear difference equations with variable coefficients, in particular to solving an interesting problem from the Putnam student competition in 1990.

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1. Introduction and preliminaries

Problems of determining a sequence or examining its convergence often appear at various student competitions when the sequence is given by a higher-order linear recurrent relation with variable coefficients. Considering that such recurrence relation is equivalent to a linear difference equation with variable coefficients, it is essential that students are prepared to solve these equations exactly. There are different methods to solve them, and we will demonstrate several methods in this paper.

The following problem appeared at the famous Putnam (USA) student competition, which motivated us to write this paper.

PROBLEM 1. (*Problem A1, Putnam, 1990* [1]) *Prove that the sequence* 2, 3, 6, 14, 40, 152, 784, ... *with general term*

$$T_n = (n+4)T_{n-1} - 4nT_{n-2} + (4n-8)T_{n-3}, \quad (n \geq 3), \tag{1}$$

is the sum of two well-known sequences.

The official solution reads:

"*Answer*: $n! + 2^n$. *Easy.*

This is not a nice problem. We know the answer is easy (because A1 is almost always easy), so we are looking for something simple. Just try substituting various simple sequences until you recognize the result. I was lucky: 152, 784 vaguely reminded me od 120, 720."

However, the problem would become much more difficult if one were asked to determine the general form of the sequence satisfying the given recursive relation. Since there are many problems of this type, we will offer four different methods for solving Problem 1. As such recursive relation is equivalent to the corresponding difference equation, the mentioned methods are methods for solving linear difference equations with variable coefficients, such as the difference equation (1). We will solve the problem with the following methods: the operator method, the invariant method, the method of generating functions, and the Z-transform method.

First, let us familiarize ourselves with the basic terms from so-called *difference calculus* (see. e.g. [2, 3, 4, 5]), which is the discrete analog of the familiar differential and integral calculus.

DEFINITION 2. Let $x(t)$ be a function of a real or complex variable t. The *difference operator* is defined by

$$\Delta x(t) = x(t+1) - x(t). \tag{2}$$

If we assume that the domain of the function x is the set $\{1, 2, \dots\}$, i.e., $t = n$, then (2) can be written as

$$\Delta x_n = x_{n+1} - x_n.$$

DEFINITION 3. The *shift operator* is defined by

$$Ex(t) = x(t+1),$$

or, in particular case, $Ex_n = x_{n+1}$.

Note that $\Delta a = 0$ and $\Delta a^t = (a-1)a^t$, where a is a constant.

DEFINITION 4. An *antidifference operator* (or *antidifference*) of $x(t)$, denoted as $\Delta^{-1}x(t)$, is any function such that, for each t,

$$\Delta(\Delta^{-1}x(t)) = x(t).$$

THEOREM 5. *If $y(t)$ is an antidifference of $x(t)$, then any antidifference of $x(t)$ is given by*

$$\Delta^{-1}x(t) = y(t) + C(t),$$

where $C(t)$ is a function with the same domain as the function x and such that $\Delta C(t) = 0$.

Assume that function x is defined on a set of the form $\{a, a+1, a+2, \dots\}$. Then, every antidifference of $x(t)$ is of the form $\Delta^{-1}x(t) = y(t) + C$, where C is a constant.

Note that $\Delta\left(\sum\limits_{k=0}^{n-1} a_k\right) = a_n$, wherefrom we obtain

$$\sum_{k=0}^{n-1} a_k = \Delta^{-1}(a_n) + C, \tag{3}$$

where C is a constant.

Also, the following so-called *summation by parts*

$$\Delta^{-1}(x_n\,\Delta y_n) = x_n y_n - \Delta^{-1}(E y_n\,\Delta x_n), \tag{4}$$

is often useful.

Since the method of generating functions will be used, it is also necessary to introduce this concept.

Definition 6. If there exists a function $g(x)$ such that

$$g(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{5}$$

for all x in a neighborhood of zero, then g is called the *generating function* for the sequence $\{a_n\}_{n=0}^{\infty}$.

The method of generating functions can sometimes be used successfully when solving linear differential equations with constant or variable coefficients. The generating function often satisfies some differential equation, which is solvable and whose solution is expressed using elementary functions. However, if this is not the case, a different tactic must be used, as will be demonstrated in the next section.

Let us now introduce the notion of Z-transform, a method similar to that of generating function.

Definition 7. The *Z-transform* of a sequence $\{x_n\}_{n=0}^{\infty}$ is a function $X(z)$ of a complex variable defined by

$$X(z) = Z[x_n] = \sum_{n=0}^{\infty} \frac{x_n}{z^n}, \tag{6}$$

and we say that Z-transform exists provided there is a number $R > 0$ such that the series in (6) converges for $|z| > R$.

It is to be expected that the inverse Z-transform should be defined simultaneously, and the main features of both transformations would be used. However, due to the specificity of our problem, just the definition of the Z-transform will suffice.

2. Four solutions for Problem 1

Solution 1. *Method of factorization operators*

The equality (1) can be written in the form

$$T_{n+3} - (n+7)T_{n+2} + 4(n+3)T_{n+1} - 4(n+1)T_n = 0, \quad n \geq 0, \tag{7}$$

which is a third-order homogeneous linear difference equation with variable coefficients; initial conditions are $T_0 = 2$, $T_1 = 3$, $T_2 = 6$. By using shift operator E, the equation (7) becomes

$$(E - A_n)(E - B_n)(E - C_n)T_n = 0, \tag{8}$$

i.e., $(E-A_n)(E-B_n)(T_{n+1}-C_nT_n)=0$, which is the same as the following

$$(E-A_n)(T_{n+2}-C_{n+1}T_{n+1}-B_nT_{n+1}+B_nC_nT_n)=0,$$

or

$$T_{n+3}-C_{n+2}T_{n+2}-B_{n+1}T_{n+2}+B_{n+1}C_{n+1}T_{n+1}-A_nT_{n+2} \\ +A_nC_{n+1}T_{n+1}+A_nB_nT_{n+1}-A_nB_nC_nT_n=0.$$

The last equation can be written as

(9) $$T_{n+3}-(A_n+B_{n+1}+C_{n+2})T_{n+2} \\ +(A_nB_n+A_nC_{n+1}+B_{n+1}C_{n+1})T_{n+1}-A_nB_nC_nT_n=0.$$

By comparing the equation (9) with (7) we get

$$A_nB_nC_n=4(n+1), \qquad A_n+B_{n+1}+C_{n+2}=n+7, \\ A_nB_n+A_nC_{n+1}+B_{n+1}C_{n+1}=4(n+3).$$

With a little effort, it can be seen that the following holds

$$A_n=2, \quad B_n=2, \quad C_n=n+1,$$

and by substituting this into (8), we have that $(E-2)(E-2)(E-(n+1)T_n)=0$, i.e.,

(10) $$(E-2)(E-2)(T_{n+1}-(n+1)T_n)=0.$$

If we set $y_n=(E-2)(T_{n+1}-(n+1)T_n)$, then from (10) it implies that

$$(E-2)y_n=0 \iff y_{n+1}-2y_n=0 \iff \frac{y_{n+1}}{2^{n+1}}-\frac{y_n}{2^n}=0 \\ \iff \Delta\Big(\frac{y_n}{2^n}\Big)=0 \iff \frac{y_n}{2^n}=C_1.$$

Since $y_0=(E-2)(T_1-1\cdot T_0)=T_2-4T_1+2T_0=-2$, we obtain that $C_1=-2$ and $y_n=-2^{n+1}$, that is

$$(E-2)(T_{n+1}-(n+1)T_n)=-2^{n+1}.$$

If $x_n=T_{n+1}-(n+1)T_n$, then from (10) we have $(E-2)x_n=-2^{n+1}$, that is

$$x_{n+1}-2x_n=-2^{n+1} \iff \frac{x_{n+1}}{2^{n+1}}-\frac{x_n}{2^n}=-1 \iff \Delta\Big(\frac{x_n}{2^n}\Big)=-1 \\ \iff \frac{x_n}{2^n}=\Delta^{-1}(-1)+C_2 \iff x_n=-n\,2^n+2^nC_2.$$

Since $x_0=T_1-T_0=1$, it implies that $C_2=1$ and

$$x_n=-n2^n+2^n \iff T_{n+1}-(n+1)T_n=-n2^n+2^n \\ \iff \frac{T_{n+1}}{(n+1)!}-\frac{T_n}{n!}=\frac{-n2^n+2^n}{(n+1)!} \iff \Delta\Big(\frac{T_n}{n!}\Big)=\frac{-n2^n+2^n}{(n+1)!} \\ \iff \frac{T_n}{n!}=\Delta^{-1}\Big(\frac{-n2^n}{(n+1)!}\Big)+\Delta^{-1}\Big(\frac{2^n}{(n+1)!}\Big)+C_3.$$

By using (4) we have that

$$\Delta^{-1}\Big(\frac{2^n(-n)}{(n+1)!}\Big) = \left\| \begin{matrix} x(n)=2^n & Ey(n)=\frac{1}{(n+1)!} \\ \Delta y(n)=\frac{-n}{(n+1)!} & y(n)=\frac{1}{n!} \end{matrix} \right\| = \frac{2^n}{n!} - \Delta^{-1}\frac{2^n}{(n+1)!}.$$

Now,

$$\frac{T_n}{n!} = \frac{2^n}{n!} - \Delta^{-1}\frac{2^n}{(n+1)!} + \Delta^{-1}\frac{2^n}{(n+1)!} + C_3 = \frac{2^n}{n!} + C_3,$$

wherefrom it follows that $\frac{T_0}{0!} = \frac{2^0}{0!} + C_3$, i.e., $C_3 = 1$. Finally, we get $\frac{T_n}{n!} = \frac{2^n}{n!} + 1$, that is $T_n = n! + 2^n$.

Solution 2. *Method of generating functions*

Let $g(x) = \sum_{n=0}^{\infty} T_n x^n$ be the generating function of the sequence $\{x_n\}_{n\in\mathbb{N}}$. Multiplying the equation (7) by $g(x)$ and summing from 0 to ∞, we get

$$\sum_{n=0}^{\infty} T_{n+3}x^n - \sum_{n=0}^{\infty}(n+7)T_{n+2}x^n + 4\sum_{n=0}^{\infty}(n+3)T_{n+1}x^n - 4\sum_{n=0}^{\infty}(n+1)T_n x^n = 0,$$

which is the same as the following equation

(11)
$$\sum_{n=0}^{\infty} T_{n+3}x^n - \sum_{n=0}^{\infty}(n+2)T_{n+2}x^n - 5\sum_{n=0}^{\infty}T_{n+2}x^n + 4\sum_{n=0}^{\infty}(n+1)T_{n+1}x^n$$
$$+ 8\sum_{n=0}^{\infty}T_{n+1}x^n - 4\sum_{n=0}^{\infty}nT_n x^n - 4\sum_{n=0}^{\infty}T_n x^n = 0.$$

Since

$$\sum_{n=0}^{\infty} T_{n+3}x^n = \frac{g(x) - T_0 - T_1 x - T_2 x^2}{x^3} = \frac{g(x) - 2 - 3x - 6x^2}{x^3},$$
$$\sum_{n=0}^{\infty} T_{n+2}x^n = \frac{g(x) - T_0 - T_1 x}{x^2} = \frac{g(x) - 2 - 3x}{x^2},$$
$$\sum_{n=0}^{\infty} T_{n+1}x^n = \frac{g(x) - T_0}{x} = \frac{g(x) - 2}{x}, \quad \sum_{n=0}^{\infty}(n+2)T_{n+2}x^n = \frac{g'(x) - T_1}{x} = \frac{g'(x) - 3}{x},$$
$$\sum_{n=0}^{\infty}(n+1)T_{n+1}x^n = g'(x), \quad \sum_{n=0}^{\infty} nT_n x^n = xg'(x),$$

the equation (11) becomes

$$\frac{g(x) - 2 - 3x - 6x^2}{x^3} - \frac{g'(x) - 3}{x} - 5\,\frac{g(x) - 2 - 3x}{x^2} + 4g'(x)$$
$$+ 8\,\frac{g(x) - 2}{x} - 4xg'(x) - 4g(x) = 0,$$

i.e.,

(12)
$$g'(x) + \frac{x-1}{x}g(x) = -\frac{1}{x^2} + \frac{3x-1}{x^2(2x-1)^2}.$$

The equation (12) is a first-order differential equation. The standard procedure would be to obtain the solution of the differential equation and then to write it as a sum of some power series, from which T_n would be calculated. However, the situation here is specific because the solution of the equation (12) cannot be expressed using elementary function. For this reason, we will replace the functions $g(x) = \sum_{n=0}^{\infty} T_n x^n$ and $g'(x) = \sum_{n=1}^{\infty} nT_n x^{n-1}$ in (12). Since

$$\frac{3x-1}{x^2(2x-1)^2} = -\frac{1}{x} - \frac{1}{x^2} + \frac{2x}{(2x-1)^2} = -\frac{1}{x} - \frac{1}{x^2} + \sum_{n=1}^{\infty} n2^{n+1}x^n,$$

the equation (12) becomes

$$\sum_{n=1}^{\infty} nT_n x^{n-1} = \Big(\frac{1}{x^2} - \frac{1}{x}\Big)\sum_{n=0}^{\infty} T_n x^n - \frac{2}{x^2} - \frac{1}{x} + \sum_{n=1}^{\infty} n2^{n+1}x^n,$$

i.e.,

$$\sum_{n=0}^{\infty} (n+1)T_{n+1}x^n = \sum_{n=0}^{\infty} T_n x^{n-2} - \sum_{n=0}^{\infty} T_n x^{n-1} - \frac{2}{x^2} - \frac{1}{x} + \sum_{n=1}^{\infty} n2^{n+1}x^n.$$

By writing down all the sums in powers of x^n, we get

$$T_1 + \sum_{n=0}^{\infty} (n+1)T_{n+1}x^n = \sum_{n=-2}^{\infty} T_{n+2}x^n - \sum_{n=-1}^{\infty} T_{n+1}x^n - \frac{2}{x^2} - \frac{1}{x} + \sum_{n=1}^{\infty} n2^{n+1}x^n,$$

i.e.,

$$\sum_{n=1}^{\infty} [(n+1)T_{n+1} - T_{n+2} + T_{n+1} - n2^{n+1}]x^n = \frac{T_0 - 2}{x^2} + \frac{T_1 - T_0 - 1}{x} + T_2 - 2T_1,$$

where $T_0 = 2$, $T_1 = 3$, and $T_2 = 6$, and wherefrom

$$(n+1)T_{n+1} - T_{n+2} + T_{n+1} - n2^{n+1} = 0, \quad T_{n+2} = (n+2)T_{n+1} - n2^{n+1},$$

that is $T_{n+2} = (n+2)T_{n+1} - n2^{n+1}$. This equation can be written in the form

$$T_{n+1} = (n+1)T_n - (n-1)2^n, \tag{13}$$

which is a first-order difference equation. Its solution is of the form

$$\begin{aligned} T_n &= \Big[\prod_{i=0}^{n-1}(i+1)\Big]T_0 - \sum_{k=0}^{n-1}\Big[\prod_{i=k+1}^{n-1}(i+1)\Big](k-1)2^k \\ &= T_0 n! + \sum_{k=0}^{n-1}(k+2)(k+3)\cdots n(1-k)2^k = 2n! + n!\sum_{k=0}^{n-1}\frac{(1-k)2^k}{(k+1)!}. \end{aligned}$$

By using (3) and the results given in Solution 1, we obtain $\sum_{k=0}^{n-1}\frac{(1-k)2^k}{(k+1)!} = \frac{2^n}{n!} + C$, so that

$$T_n = 2n! + n!\Big(\frac{2^n}{n!} + C\Big).$$

Using the initial conditions for $n = 1$ we get $C = -1$, thus $T_n = n! + 2^n$. The solution of the differential equation (12) will be $g(x) = \sum_{n=0}^{\infty} (n! + 2^n)x^n$, and considering the introduced substitution $g(x) = \sum_{n=0}^{\infty} T_n x^n$ gives $T_n = n! + 2^n$, which is the solution to Problem 1.

REMARK 8. As we have already mentioned, when we use the method of generating functions, we often obtain the solution of the differential equation, which can be written as the sum of some powers series, from which T_n (or x_n) would be calculated. So, for example, solving the equation [4, Problem 3.3.37]

$$x_{n+2} = \frac{3}{n+2} x_{n+1} - \frac{2}{(n+1)(n+2)} x_n, \quad n = 0, 1, \ldots,$$

using the method of generating functions like the above yields the following differential equation

$$g''(t) - 3g'(t) + 2g(t) = 0,$$

whose solution is

$$\sum_{n=0}^{\infty} x_n t^n = g(x) = C_1 e^t + C_2 e^{2t} = \sum_{n=0}^{\infty} \frac{1}{n!} (C_1 + C_2 2^n) t^n,$$

wherefrom it follows that $x_n = \frac{1}{n!}(C_1 + C_2 2^n)$, $n = 0, 1, \ldots$.

SOLUTION 3. *Method of invariants*

Equation (7) can be transformed to the following form

$$T_{n+3} - (n+5)T_{n+2} + 2(n+2)T_{n+1} = 2[T_{n+2} - (n+4)T_{n+1} + 2(n+1)T_n],$$

from which

$$\begin{aligned}
&\frac{1}{2^{n+2}}[T_{n+3} - (n+5)T_{n+2} + 2(n+2)T_{n+1}] \\
&\quad = \frac{1}{2^{n+1}}[T_{n+2} - (n+4)T_{n+1} + 2(n+1)T_n] = \cdots = \frac{1}{2}[T_2 - 4T_1 + 2T_0] = -1.
\end{aligned}$$

That is how we get

$$T_{n+2} - (n+4)T_{n+1} + 2(n+1)T_n = -2^{n+1}, \tag{14}$$

which is one invariant of the initial equation. Equation (14) is a second-order equation, and in this way, we managed to lower the order of the difference equation by one. Now, let us try to find the invariant of the equation (14). Namely, the equation (14) can be written in the form

$$\begin{aligned}
\frac{1}{2^{n+1}}[T_{n+2} - (n+2)T_{n+1}] + n &= \frac{1}{2^n}[T_{n+1} - (n+1)T_n] + n - 1 \\
&= \cdots = \frac{1}{2^0}[T_1 - T_0] - 1 = 0.
\end{aligned}$$

This implies that $\frac{1}{2^n}[T_{n+1} - (n+1)T_n] + n - 1 = 0$, i.e.,

$$T_{n+1} = (n+1)T_n + (1-n)2^n. \tag{15}$$

Equation (15) is a first-order non-homogeneous linear difference equation. Using the method of invariants, we managed to reduce the third-order difference equation to a first-order one. Since the equations (13) and (15) are the same, its solution is $T_n = n! + 2^n$, too.

SOLUTION 4. *Z-transform method*

By applying Z-transform to the equation (1), we get

$$\begin{aligned} Z[T_{n+3}] - Z[(n+2)T_{n+2}] - 5Z[T_{n+2}] + 4Z[(n+3)T_{n+1}] \\ + 8Z[T_{n+1}] - 4Z[nT_n] - 4Z[T_n] = 0. \end{aligned} \tag{16}$$

Since,

$$\begin{aligned} Z[T_n] &= \sum_{n=0}^{\infty} \frac{T_n}{z^n} = X(z), \quad Z[nT_n] = -zX'(z), \quad Z[T_{n+1}] = zX(z) - 2z, \\ Z[(n+1)T_{n+1}] &= \sum_{n=0}^{\infty} \frac{(n+1)T_{n+1}}{z^n} = z\sum_{n=1}^{\infty} \frac{nT_n}{z^n} = z(-zX'(z)), \\ Z[T_{n+2}] &= z^2(Z[T_n] - T_0 - T_1 z) = z^2X(z) - 2z^2 - 3z^3, \\ Z[(n+2)T_{n+2}] &= \sum_{n=0}^{\infty} \frac{(n+2)T_{n+2}}{z^n} = \sum_{n=2}^{\infty} \frac{nT_n}{z^{n-2}} = z^2\sum_{n=2}^{\infty} \frac{nT_n}{z^n} = z^2\Big(\sum_{n=1}^{\infty} \frac{nT_n}{z^n} - \frac{T_1}{z}\Big) \\ &= z^2\Big(\sum_{n=1}^{\infty} \frac{nT_n}{z^n} - \frac{3}{z}\Big) = z^2\Big(-zX'(z) - \frac{3}{z}\Big) = -z^3X'(z) - 3z, \\ Z[T_{n+3}] &= z^3\Big(Z[T_n] - T_0 - \frac{T_1}{z} - \frac{T_2}{z^2}\Big) = z^3X(z) - 2z^3 - 3z^2 - 6z, \end{aligned}$$

by substituting these expressions into (16), we obtain

$$\begin{aligned} &z^3X(z) - 2z^3 - 3z^2 - 6z + z^3X'(z) + 3z - 5(z^2X(z) - 2z^2 - 3z^3) \\ &\quad + 4(-z^2X'(z) + 2zX(z) - 4z) + 8(zX(z) - 2z) + 4zX'(z) - 4X(z) = 0, \end{aligned}$$

i.e., after rearranging,

$$X'(z) + \Big(1 - \frac{1}{z}\Big)X(z) - \frac{2z^2}{(z-2)^2} + \frac{7z}{(z-2)^2} - \frac{4}{(z-2)^2} = 0. \tag{17}$$

This equation is a first-order differential equation. The usual procedure is to obtain the differential equation's solution and calculate the inverse Z-transform. However, the situation here is specific because the solution of the equation (17) cannot be expressed using elementary functions. Therefore, we will proceed as in Solution 2.

We have

$$X(z) = \sum_{n=0}^{\infty} \frac{T_n}{z^n}, \quad X'(z) = -\sum_{n=1}^{\infty} \frac{nT_n}{z^{n+1}} = -\sum_{n=2}^{\infty} \frac{(n-1)T_{n-1}}{z^n},$$

and by standard procedures we obtain

$$\frac{2z^2}{(z-2)^2}=\sum_{n=0}^{\infty}\frac{(n+1)2^n}{z^n},\qquad \frac{z}{(z-2)^2}=\sum_{n=1}^{\infty}\frac{n2^{n-1}}{z^n},$$
$$\frac{1}{z-2}=\sum_{n=1}^{\infty}\frac{2^{n-1}}{z^n},\qquad \frac{1}{(z-2)^2}=\sum_{n=2}^{\infty}\frac{(n-1)2^{n-2}}{z^n}.$$

Substituting these expressions into (17), we have

$$-\sum_{n=2}^{\infty}\frac{(n-1)T_{n-1}}{z^n}+\sum_{n=0}^{\infty}\frac{T_n}{z^n}-\frac{1}{z}\sum_{n=0}^{\infty}\frac{T_n}{z^n}$$
$$-2\sum_{n=0}^{\infty}\frac{(n+1)2^n}{z^n}+7\sum_{n=1}^{\infty}\frac{n2^{n-1}}{z^n}-4\sum_{n=2}^{\infty}\frac{(n-1)2^{n-2}}{z^n}=0,$$

i.e.,

$$\sum_{n=2}^{\infty}\frac{-(n-1)T_{n-1}+T_n-T_{n-1}-(n+1)2^{n+1}+7n2^{n-1}-4(n-1)2^{n-2}}{z^n}=0,$$

which implies

$$T_{n+1}=(n+1)T_n+(1-n)2^n.$$

This equation is the same as the equation (13), and we obtain that $T_n=n!+2^n$.

Remark 9. A somewhat more complicated problem would be if, instead of solving the equation (1), we were asked to examine the convergence of the following sequences: $\{T_n\}_{n=0}^{\infty}$, $\left\{\frac{T_n}{n!}\right\}_{n=0}^{\infty}$, and $\left\{\frac{T_n}{2^n}\right\}_{n=0}^{\infty}$. By solving the equation (1) using some of the methods mentioned above, we get

a) $\lim\limits_{n\to\infty}T_n=\lim\limits_{n\to\infty}(n!+2^n)=+\infty$, the sequence $\{T_n\}_{n=0}^{\infty}$ is divergent,

b) $\lim\limits_{n\to\infty}\frac{T_n}{n!}=\lim\limits_{n\to\infty}\left(1+\frac{2^n}{n!}\right)=1$, the sequence $\left\{\frac{T_n}{n!}\right\}_{n=0}^{\infty}$ is convergent,

c) $\lim\limits_{n\to\infty}\frac{T_n}{2^n}=\lim\limits_{n\to\infty}\left(1+\frac{n!}{2^n}\right)=+\infty$, the sequence $\left\{\frac{T_n}{2^n}\right\}_{n=0}^{\infty}$ is divergent.

3. About another method

Although it cannot be helpful when solving Problem 1, it is still important to familiarize us with another method for solving linear difference equations with variable coefficients, the so-called factorial series method. For this purpose, we introduce *falling factorial power* according to the following definition.

Definition 10. The *falling factorial power* $t^{(s)}$ is defined as follows, according to the value of s.

1. If s is a positive integer, then $t^{(s)}=t\,(t-1)\cdots(t-s+1)$.
2. If $s=0$, then $t^{(0)}=1$.
3. If s is a negative integer, then $t^{(s)}=\dfrac{1}{(t+1)\,(t+2)\cdots(t-s)}$.

4. If s is not an integer, then $t^{(s)} = \dfrac{\Gamma(t+1)}{\Gamma(t-s+1)}$, where Γ is the well-known gamma function.

Let us highlight one significant fact that is used in the factorial series method:

$$\Delta_t t^{(s)} = st^{(s-1)}, \quad s \neq 0.$$

The following example will illustrate the application of the factorial series method.

EXAMPLE 11. Enquire with respect to the convergence the sequence given by the following recurrence relation:

$$x_{n+2} = \frac{3(n+2)}{n+3} x_{n+1} - \frac{2n+1}{n+3} x_n, \quad n = 1, 2, \ldots \tag{18}$$

Solution. The equation (18) is equivalent to the following linear equation

$$(n+3)x_{n+2} - 3(n+2)x_{n+1} + (2n+1)x_n = 0, \quad n = 1, 2, \ldots . \tag{19}$$

Equation (19) is a homogeneous second-order linear difference equation whose solution is the general term of the given sequence. To determine the solution of this equation, we will use the method of descending factorials series after translating it into the appropriate form, as follows

$$(n+3)E^2 x_n - 3(n+2)Ex_n + (2n+1)x_n = 0.$$

Since $E = \triangle + I$, we obtain

$$(n+3)(\Delta + I)^2 x_n - 3(n+2)(\Delta + I)x_n + (2n+1)x_n = 0,$$

i.e.,

$$(n+3)\Delta^2 x_n - n\Delta x_n - 2x_n = 0. \tag{20}$$

Assume that the solution x_n has the form $x_n = \sum\limits_{k=-\infty}^{\infty} c_k n^{(k)}$, where $c_k = 0$, for $k < 0$. Now, we have that

$$\Delta x_n = \sum_{k=-\infty}^{\infty} kc_k n^{(k-1)}, \quad \Delta^2 x_n = \sum_{k=-\infty}^{\infty} k(k-1)c_k n^{(k-2)}. \tag{21}$$

By substituting (21) into (20), we get

$$\begin{aligned} \sum_{k=-\infty}^{\infty} k(k-1)c_k nn^{(k-2)} &+ \sum_{k=-\infty}^{\infty} 3k(k-1)c_k n^{(k-2)} \\ &- \sum_{k=-\infty}^{\infty} kc_k nn^{(k-1)} - \sum_{k=-\infty}^{\infty} 2c_k n^{(k)} = 0. \end{aligned} \tag{22}$$

Since $nn^{(m)} = n^{(m+1)} + mn^{(m)}$, (22) can be written in the following form

$$\begin{aligned} \sum_{k=-\infty}^{\infty} k(k-1)c_k[n^{(k-1)} + (k-2)n^{(k-2)}] &+ \sum_{k=-\infty}^{\infty} 3k(k-1)c_k n^{(k-2)} \\ - \sum_{k=-\infty}^{\infty} kc_k[n^{(k)} + (k-1)n^{(k-1)}] &- \sum_{k=-\infty}^{\infty} 2c_k n^{(k)} = 0. \end{aligned}$$

By changing the index in the previous sums, we get

$$\sum_{k=-\infty}^{\infty}\{(k+1)(k+2)(k+3)c_{k+2}-(k+2)c_k\}n^{(k)}=0. \tag{23}$$

Since (23) is an identity, each coefficient must be zero, i.e.,

$$(k+1)(k+2)(k+3)c_{k+2}-(k+2)c_k=0, \quad k=0,1,2,\dots,$$

or $(k+1)(k+3)c_{k+2}-c_k=0$, $k=0,1,2,\dots$. From this, we have

$$1\cdot 3\cdot C_2-C_0=0,\ 2\cdot 4\cdot C_3-C_1=0,\ 3\cdot 5\cdot C_4-C_2=0,\ 4\cdot 6\cdot C_5-C_3=0,\ \dots,$$

i.e.,

$$C_2=\frac{C_0}{1\cdot 3},\quad C_3=\frac{C_1}{2\cdot 4},\quad C_4=\frac{C_0}{1\cdot 3^2\cdot 5},\quad C_5=\frac{C_1}{2\cdot 4^2\cdot 6},\quad \dots,$$

which implies that the solution of the equation (18) has the form

$$x_n=C_0\left(1+\frac{n^{(2)}}{1\cdot 3}+\frac{n^{(4)}}{1\cdot 3^2\cdot 5}+\frac{n^{(6)}}{1\cdot 3^2\cdot 5^2\cdot 7}+\cdots\right) + C_1\left(n^{(1)}+\frac{n^{(3)}}{2\cdot 4}+\frac{n^{(5)}}{2\cdot 4^2\cdot 6}+\frac{n^{(7)}}{2\cdot 4^2\cdot 6^2\cdot 8}+\cdots\right),$$

and $\lim_{n\to\infty} x_n=+\infty$. It means that the sequence is divergent.

4. Conclusion

Motivated by Problem 1, we demonstrated several methods for solving linear difference equations with variable coefficients: the operator method, the method of invariants, the method of generating functions, the Z-transform method, and the factorial series method. In doing so, a particular curiosity emerged when using the methods of generating functions and the Z-transform method. Namely, an unusual situation occurred when the solutions of the differential equations obtained by these methods could not be represented using elementary functions. That is why we had to solve the differential equations using the power series method and, in this way, finally get the explicit form of the general term of the given sequence. We demonstrated the factorial series method on a separate equation since it could not be applied to solving Problem 1.

As we have seen, only some methods are suitable for certain situations. That is why it is vital that students, especially those who participate in math competitions, are familiar with all the methods of solving linear difference equations with variable coefficients.

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M.N.: Department of Mathematics, University of Tuzla, U. Vejzagića 4, 75000 Tuzla, Bosnia and Herzegovina

ORCID: 0000-0003-0202-0390

E-mail: mehmed.nurkanovic@untz.ba

M.T.: Agricultural and Medical School, Brčko District, Bosnia and Herzegovina

ORCID: 0000-0002-9834-7103

E-mail: trumicmirsad@yahoo.com

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